NUMERICAL TREATMENT OF SINGULARLY PERTURBED TWO POINT BOUNDARY VALUE PROBLEMS

by

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NUMERICAL TREATMENT OF SINGULARLY PERTURBED TWO POINT BOUNDARY VALUE PROBLEMS

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Dedicated to

Sri Manikeshwari Matha

CERTIFICATE

This is to certify that the research work embodied in the thesis entitled 'Numerical Treatment of Singularly Perturbed Two Point Boundary Value Problems' by Yanala Narsimha Reddy has been carried out under my supervision and that this work has not been submitted elsewhere for the award of any degree or diploma.

April 1986

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April 1986 *Yanala Narsimha Reddy*

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LIST OF TABLES

LIST OF PUBLICATIONS

A part of the thesis has been accepted/submitted for publication in the form of following research papers

1. The Method of Inner Boundary Condition : A New Approach for Solving Singular Perturbation Problems, Journal of Computational Physics, 62 (1986), 349-360.

2. Numerical Integration of a Class of Singular Perturbation Problems, Journal of Optimization Theory and Applications, to appear.

3. Numerical Treatment of Singularly Perturbed Two Point Boundary Value Problems, Applied Mathematics and Computation, to appear.

4. Numerical Solution of Singular Perturbation Problems by Terminal Boundary Value Technique, Journal of Optimization Theory and Applications, to appear.

5. Approximate Method for the Numerical Solution of Singular Perturbation Problems, Applied Mathematics and Computation, to appear.

6. An Initial Value Technique for a Class of Nonlinear Singular Perturbation Problems, Journal of Optimization Theory and Applications, to appear.

7. Numerical Solution of Singular Perturbation Problems via Deviating Arguments, Applied Mathematics and Computation, to appear.

8. A Nonasymptotic Method for Singular Perturbation Problems, submitted.

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10. An Approximate Method for Solving a Class of Singular Perturbation Problems, submitted.

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SYNOPSIS

of the

Ph.D. Thesis

on

Numerical Treatment of Singularly Perturbed Two Point Boundary Value Problems

by

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In this thesis, we have proposed and illustrated some efficient and simple numerical techniques for solving singularly perturbed two point boundary value problems in ordinary differential equations.

Chapter 1 presents introduction, a brief survey of the asymptotic and numerical analysis of singular perturbation problems and summary of the work included in the present thesis.

In Chapter 2, an approximate method for the numerical integration of a class of linear singularly perturbed two point boundary value problems with a boundary layer on the left end of the underlying interval is presented. This method does not depend on asymptotic expansions. The original second order differential equation is replaced by an approximate first order differential equation of neutral type with a small deviating argument. Then, the trapezoidal formula is used to obtain the

three-term recurrence relationship. Discrete invariant imbedding algorithm is used to solve the tridiagonal algebraic system. The stability of this algorithm is investigated. The proposed method is iterative on the deviating argument. Some numerical examples have been solved to demonstrate the applicability of the method.

A nonasymptotic method for solving linear singularly perturbed two point boundary value problems with a boundary layer on the left end of the interval is presented in Chapter 3. This is an alternative method to that of Chapter 2, and also simpler than that method. The method is distinguished by the following fact : The original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument, and solved efficiently by employing the Simpson's rule coupled with discrete invariant imbedding algorithm. The proposed method is iterative on the deviating argument. Some numerical experiments have been included to demonstrate the efficiency of the method.

In order to know the behavior of the solution of a singular perturbation problem it is always suggestive to divide the problem into two problems and to solve them separately. Keeping this in mind, a new approach based on the method of inner boundary condition is presented in Chapter 4. The method is distinguished by the following facts : The original problem is partitioned into inner and outer region differential equation

systems. Asymptotic expansion is used to obtain the terminal boundary condition. Using an appropriate transformation, a new inner region problem is obtained and solved as a two point boundary value problem. The derivative boundary condition at the terminal point is then derived from the solution of the inner region problem. Using this condition, the outer region problem is efficiently solved by employing the classical second order central finite difference scheme. The proposed method is iterative on the terminal point of the inner region. Some test examples have been solved to illustrate the method and the computational results are compared with exact solutions.

In Chapter 5, a boundary layer technique is presented for a class of linear singular perturbation problems. It is motivated by the asymptotic behavior of the singular perturbation problem. As in Chapter 4, the original problem is divided into inner and outer region problems. However, asymptotic expansions are not employed. The reduced problem is solved to obtain the terminal boundary condition. Then, a new inner region problem is created and solved as a two point boundary value problem. In turn, the outer region problem is also modified and the resulting problem is efficiently solved by employing the trapezoidal formula coupled with discrete invariant imbedding algorithm. This technique is also iterative on the terminal point of the inner region. Numerical experience with three examples is reported.

In Chapter 6, two terminal boundary value techniques are presented for linear singular perturbation problems. These involve a modification of the singular perturbation problem so that any suitable existing standard numerical method can be employed on the modified problem. As usual, the original problem is divided into outer and inner region problems. Two techniques are introduced to obtain the terminal boundary condition in the implicit form. Then, the outer region problem is solved as a two point boundary value problem and an explicit terminal boundary condition is obtained. In turn, the inner region problem is modified and solved as a two point boundary value problem using the explicit terminal boundary condition. Three numerical examples are included to illustrate these techniques.

An approximate method for solving a class of singular perturbation problems is presented in Chapter 7. It is designed on the basis of the asymptotic behavior of the singular perturbation problem. As with other methods, the given region is divided into inner and outer regions. The original second order problem is replaced by an asymptotically equivalent first order problem and solved as an initial value problem in the inner region. A terminal boundary condition is then obtained from the solution of the inner region problem. In turn, an outer region problem is obtained, by replacing the second order differential equation by an approximate first order differential equation with a small deviating argument, and solved efficiently by employing the trapezoidal formula coupled with discrete

invariant imbedding algorithm. Several numerical examples have been solved to demonstrate the applicability of the method. Finally the method is extended to a more general class of problems. Again one numerical example is solved in this general class.

In Chapter 8, an initial value technique, which is simple to use and easy to implement, is presented for a class of nonlinear singular perturbation problems with a boundary layer on the left end of the interval. It is distinguished by the following fact : The original second order problem is replaced by an asymptotically equivalent first order problem and solved as an initial value problem. Numerical experience with several examples is described.

A boundary value method for a class of nonlinear singular perturbation problems is presented in Chapter 9. By constructing a modified problem with a boundary layer correction, a discussion on how to deal with the boundary layer separately is presented. The proposed method is iterative on the terminal point of the boundary layer region. Several numerical examples are discussed to illustrate the method. Finally, a lower bound for the terminal point of the boundary layer region is obtained in terms of the perturbation parameter.

Finally in Chapter 10, the nonasymptotic method developed in Chapter 3 has been extended for solving general linear singularly perturbed two point boundary value problems.

Firstly, problems with a right end boundary layer have been discussed. Secondly, problems with an interior layer have been discussed. Finally, problems with two boundary layers have been discussed. Numerical experience with the method for some model problems is also reported to confirm the theoretical analysis.

In a nut-shell, the numerical methods presented in this thesis for solving singularly perturbed two point boundary value problems in ordinary differential equations have been shown to be efficient over the conventional methods. Above all, these methods are conceptually simple, easy to use, and are readily adapted for computer implementation with a modest amount of problem preparation. Several model linear and nonlinear problems have been solved and numerical results are presented in respective chapters. It is observed that the accuracy predicted can always be achieved with very little computational effort. All the numerical results, presented in this thesis, have been computed on DEC-1090 computer system at IIT Kanpur.

CHAPTER 1

INTRODUCTION

1.1 GENERAL

The academic efforts approach more and more the engineering interests of methods for solving real-life problems such as the Navier-Stokes equations with large Reynolds number. The Navier-Stokes equations, thanks to the Reynolds number, had become one of the most striking examples of singular perturbations, leading to the idea of boundary layer, due to Prandtl. Prandtl's theory (developed later on in a more rigorous and consistent way) provides a beautiful mathematical tool for the investigation of practical problems. Still, it is unable to explain some other phenomena, such as the appearance of turbulent flows (a well known boundary layer paradox). Hence, there was a division among the fluid dynamicists. In the book by Garrett Birkhoff : "Hydrodynamics a study in logic, fact and similitude", one comes across the following witticism "fluid dynamicists were divided into hydraulic engineers who observed what could not be explained, and mathematicians who explained things that could not be observed". The developments of the small parameter methods led to its efficient use in other fields of applied mathematics. Here, one can mention, for instance : fluid mechanics, fluid dynamics, elasticity, quantum mechanics, plasticity, chemical reactor theory, aerodynamics, plasma dynamics, magneto hydro dynamics, rarefied-gas dynamics, oceanography, meteorology,

diffraction theory, reaction-diffusion process, non-equilibrium and radiating flows and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modelling of steady and unsteady viscous flow problems with large Reynolds number, convective heat transport problems with large Peclet number etc. The problems to be treated by means of asymptotic analysis and numerical analysis are becoming progressively, more and more complicated. A new turning point in the development of asymptotic analysis occured with the introduction of numerical analysis to singular perturbation problems. This combination turns out to be an efficient mathematical tool to investigate asymptotic phenomena in applied sciences. It is done, a second small parameter h took its place next to the big brother ε.

The object of this chapter is to present a brief survey of the asymptotic and numerical analysis of singular perturbation problems. The basic purpose of this study is to find out what problems are treated and what asymptotic/numerical methods are employed, with an eye toward the goal of developing efficient and simple numerical techniques to solve singular perturbation problems. A summary of some recent methods is presented, and this leads to conclusions and recommendations about what methods to use on singular perturbation problems. Finally, some areas of current research are indicated, and summary of the work included in this thesis is presented.

1.2 ASYMPTOTIC ANALYSIS OF SINGULAR PERTURBATION PROBLEMS

Singular Perturbations, now (today), is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science and engineering. Though the basic intuitive ideas involving local patching of solutions can be found in early work by Laplace, Kirchhoff and others, Prandtl's [124] paper at the 1904 Leipzig Mathematical congress began the study of the fluid dynamical boundary layers by analyzing viscous incompressible flow past an object as the Reynolds number (R) becomes infinite. The distinguishing feature of 'Singular Perturbation Problem' occurs viz. a thin region near the solid boundary where the velocity changes from zero (as required by the no slip condition) to an outer flow which is essentially inviscid. Expressed mathematically the solution converges nonuniformly in the domain as a parameter $\varepsilon = \frac{1}{R}$ tends to zero. In the interests of clarity, we give briefly the definition of a singular perturbation problem in its simplest and most widely used form. Consider a two point boundary value problem P_ε depending on a small positive parameter ε $(0 < \varepsilon << 1)$. Under some conditions a solution $y_\varepsilon(x)$ of P_ε can be constructed by the well known method of perturbation — i.e. as a power series in ε with first term y_0 being the solution of the problem P_0. Under the happiest circumstances, this perturbation method leads to altogether satisfactory results. This series cannot often be presumed to uniformly converge, particularly for small values of ε, in the

entire interval. When such an expansion converges as $\varepsilon \to 0$, uniformly in x, one speaks then of a 'Regular Perturbation Problem'. On the otherhand, when $y_\varepsilon(x)$ does not have a uniform limit in x as $\varepsilon \to 0$, this straight forward perturbation method fails and as a consequence of this nonuniformity one may miscalculate or even lose essential results, one then speaks of a 'Singular Perturbation Problem'. A singular perturbation problem is best defined as one in which no single asymptotic expansion is uniformly valid throughout the interval, as $\varepsilon \to 0$. The prototype of singular perturbation problem is Prandtl's boundary layer theory. By definition, the boundary layer is a narrow region where the solution of a differential equation changes rapidly and the thickness of the boundary layer must approach zero as ε (small positive dimensionless parameter, $0 < \varepsilon << 1$) tends to zero. In a recent book review, O'Malley [115] gives an erudite outline of the history of singular perturbations, starting from Prandtl's [124] paper on fluid dynamical boundary layers. This article (O'Malley [115]) will remain as one of the most readable sources on asymptotic methods for singular perturbation problems. In the first part of the 20th century, analysis of asymptotic solutions to linear ordinary differential equations progressed through the work of Birkhoff [24], Langer [90], and others, with significant work on turning point problems being done by physicists Wentzel [147], Krammers [86], Brillouin [28] and others. Friedrichs and his student Wasow seem to be the first mathematicians to initiate

a prolonged study of the asymptotic solution of singularly perturbed boundary value problems. Their work was motivated by an analysis of the edge effect for buckled plates; they first used the term 'Singular Perturbations' in print in the title of Friedrichs and Wasow [58]. Other mathematicians, including Levinson [93], Tikhonov [137] and Vasileva and Volosov [139], began studying related problems soon afterwards. Levinson began a study of wide spectrum of important topics in singular perturbations and made intuitive contributions to singular perturbations (before the mid-1950's) together with a number of promising students and young collaborators including Coddington, Davis, Flatto, Haber and Levin. The Russian school also did outstanding work on many subjects including boundary layer methods — Vishik and Lyusternik [144]. From around 1950, fluid dynamicists solved some very interesting physical problems like the linoleum-rolling problem — Carrier [29] and low Reynolds number flow past bodies - Kaplaun [79,80]. At Caltech's Guggenheim Aeronautical Laboratory, Lagerstrom, Cole, Latta , Van Dyke, Kaplaun and others became equally involved in asymptotic expansion procedures for more general singular perturbation problems. A (over-simplified) matching procedure was presented in the book of Van Dyke [138]. The straight forward recipe he provided made it easy for tremendous variety of scientists to learn the rudiments of matching and to the important problems in their own disciplines. The basic idea, much as in Friedrichs early and Erdeyli [51] lectures,

involved an asymptotic matching of the inner and outer expansions at the edge of the boundary layer, where they should both be appropriate. Cole [34] stressed the limit process expansions and two timings in a context far broader than fluid mechanics. Indeed the results obtained through matching generally coincided with those known through the intuitive folkways of the various fields. Wasow [145] placed the singular perturbations in the contexts of the analytic theory of differential equations. By 1970, courses in perturbation methods became common in science, engineering and applied mathematics departments, and inevitably a string of text books and high level monographs began to appear. They include, O'Malley [113], Nayfeh [106,107], Van Dyke [138], Bender and Orszag [22], Kevorkian and Cole [84], Bellman [18], Bellman and Cooke [20], Eckhaus [43,44], Eckhaus and de Jager [45], Miranker [103], Doolan et al. [38], Kaplaun [80], Erdelyi [50], Dingle [37], Carrier and Pearson [30], Ames [3], Na [105], El'sgol'ts and Norkin [48], Driver [42], Aziz [14], Willoughby [148], Ardema [5], Verhlust [142,143], Childs et al. [31], Hughes [75], Meyer and Parter [97], Brauner et al. [27], Hemker and Miller [66], Axelsson et al. [12], Miller [100,101,102].

1.3 NUMERICAL ANALYSIS OF SINGULAR PERTURBATION PROBLEMS

As we mentioned, Numerical analysis and Asymptotic analysis are two principal approaches to singular perturbation problems. It is a little surprising that there has not been more interaction between these approaches. In our opinion, this is because the goals and the problem class are rather different. At the risk

of gross over-simplification, we would say that numerical analysis tries to provide quantitative information about a particular problem, whereas asymptotic analysis **tries** to gain insight about the qualitative behavior of a family of problems and only semiquantitative information about any particular member of the family. Numerical methods are intended for broad class of problems and are intended to minimize demands upon the problem solver. Asymptotic methods treat comparatively restricted class of problems and require the problem solver to have some understanding of the behavior of the solutions expected. To the extent that the goals are different, the approaches do not overlap. When they do overlap, they tend to be complementary. The problems and methods of singular perturbation theory are of unquestionable value in this area right now, and they hold great promise for the future. As soon as an asymptotic analysis is valid and a few terms in the asymptotic expansion describe the solution sufficiently accurate, one usually can rely on standard techniques to obtain the solutions. As soon as an asymptotic analysis is difficult to handle or perform badly, one usually asks for numerical analysis for solving singular perturbation problems. However, the numerical analysis of singular perturbation problems mainly concentrates on the following question : how to find a numerical approximation to the solution for small values of the parameter $\varepsilon(=\frac{1}{R})$, where no short asymptotic expansion is available. Numerical treatment of singular perturbation problems has always been far from trivial,

because of the boundary layer behavior of the solutions. However, the area of singular perturbations is a field of increasing interest to applied mathematicians. Pearson [122] was perhaps the first to attempt something like net adjustments in difference schemes while treating singular perturbation problems. Basically, his idea was to use a variable mesh width. Based on the classical three-point difference scheme for a non-uniform mesh, Pearson gives the numerical solution of a great variety of singular perturbation problems of the form

$$L_\varepsilon[y] = \varepsilon\, y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{1.1}$$

$$\text{for } -1 \le x \le 1 \text{ with } y(-1) = A \text{ and } y(1) = B \tag{1.2}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$). He used a simple variable mesh approximation

$$\frac{2\varepsilon\,[q_i y_{i+1} + p_i y_{i-1} - (p_i+q_i)y_i]}{p_i q_i (p_i+q_i)} + a(x_i)\,\frac{[q_i^2 y_{i+1} - p_i^2 y_{i-1} + (p_i^2 - q_i^2)y_i]}{p_i q_i (p_i+q_i)}$$

$$+\, b(x_i)y_i = f(x_i) \tag{1.3}$$

where $p_i = x_{i+1} - x_i$ and $q_i = x_i - x_{i-1}$. This is a first order approximation, if we assume the mesh ratio q_i/p_i as constant. A difficult solution procedure is followed, which finally requires a few thousand mesh points in the interval $[-1,1]$. The mesh must be properly chosen so that the solution of the difference equation approximates that of the differential equation. This is accomplished by iteratively adjusting the mesh spacing

such that the mesh points are concentrated in the regions where $y(x)$ changes rapidly. The procedure is first to solve the problem with a uniform mesh and a modest value of ε (i.e., $\varepsilon = 10^{-1}$ or 10^{-2}). Then insert new mesh points between adjacent points, say x_i and x_{i+1}, for which a certain predetermined tolerance is exceeded (e.g., $|y_{i+1}-y_i| > \delta$, δ prescribed). A smoothing process is carried out to avoid locally abrupt changes in the mesh intervals. Next the problem is solved by Gaussian elimination. Finally, the size of ε is decreased and the procedure is repeated, using the mesh obtained from the preceding ε-step as the initial mesh. Obviously, this method is costly in terms of computer time even for simple linear problems. Pearson [123] has also given the solution of the nonlinear problem

$$F(x, y, y', \varepsilon y'') = 0 \tag{1.4}$$

for $0 \leq x \leq 1$ with $y(0) = \alpha$ and $y(1) = \beta$ (1.5)

The same approximations as in (1.3) are used for y' and y''. The resulting nonlinear algebraic equations were solved by the Newton-Raphson iterative scheme. For example, to solve the nonlinear problem

$$\varepsilon y'' + (y')^2 = 1 \tag{1.6}$$

$$y(0) = y(1) = 1 \tag{1.7}$$

where $\varepsilon = 10^{-6}$, Pearson used 4000 mesh points to get a solution which agreed up to five significant digits with the exact solution.

In a more difficult problem, he used 25000 mesh points to get an accurate solution. In an attempt to overcome the numerical instability of the standard methods the upstream (upwind) one-sided (directional) differences on a uniform mesh was introduced. For detailed discussion see Dorr [39]. The essential idea of this type of scheme is to replace the first order derivatives in (1.1) by a one-sided difference quotient instead of the centered difference. The choice of a backward or forward difference depends on the sign of a(x) at the particular mesh point x_i under consideration, that is

$$\delta[ay_i] = \begin{cases} a(x_i)(y_{i+1}-y_i)/h & \text{if } a(x_i) \geq 0 \\ a(x_i)(y_i-y_{i-1})/h & \text{if } a(x_i) < 0 \end{cases}$$

Then the difference equation takes the form

$$L_h[y_i] = \varepsilon\left[\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right] + \delta[ay_i] + b(x_i)y_i = f_i \qquad (1.8)$$

If $b(x) \leq 0$ then it can be shown that L_h is a difference operator of positive type and, hence there exists a unique solution for each set of given data and for each $\varepsilon > 0$, $h > 0$. By a difference operator of positive type we mean :

Definition : A difference operator L_h of the form

$$L_h[y_i] = A_i y_{i-1} + B_i y_i + C_i y_{i+1}$$

is of <u>positive type</u> if

(i) $A_i > 0$, $C_i > 0$, $\forall$ i, and (ii) $A_i+B_i+C_i \leq 0$, $\forall$ i.

With these above restrictions on $b(x)$, L_h satisfies a discrete maximum principle. Moreover, if the directional difference method is used, then for fixed $h > 0$ there is a limit function, as ε tends to zero, which satisfies the 'reduced' difference equation, (1.8) with $\varepsilon = 0$. One drawback of this method is that it is only first order. The idea was developed further by Dorr et al. [41], and they discussed the applications of the maximum principle to obtain elementary estimates for solutions of second order ordinary differential equations. These estimates are applied to obtain results on the limiting behavior of solutions of singular perturbation problems. They considered the linear problems under various hypotheses, including turning point problems, and quasilinear problems of the form

$$\varepsilon y''(t) + \alpha(t, y(t), \varepsilon) y'(t) = \beta(t, y(t), \varepsilon) \tag{1.9}$$

for $a < t < b$ with $y(a) = A(\varepsilon)$ and $y(b) = B(\varepsilon)$ (1.10)

Il'in [76] constructs a difference scheme which represents the rate of decay in the boundary layer correctly for the homogeneous case of (1.1) with constant coefficients and $b = 0$. Then the scheme is applied to the more general equation

$$\varepsilon y'' + a(x) y' = f(x) \tag{1.11a}$$

and for a uniform mesh the following difference operator is obtained

$$L_h[y_i] = \gamma_i \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right] + a(x_i) \left[\frac{y_{i+1} - y_{i-1}}{2h} \right] = f(x_i) \quad (1.11b)$$

where γ_i is chosen so that the scheme is $O(h^2)$ accurate and correctly represents the rate of decay from the boundary layer into the interior. Il'in found for equation (1.11a) that

$$\gamma_i = \frac{a(x_i)h}{2} \cot h \left[\frac{a(x_i)h}{2\varepsilon} \right] \quad (1.11c)$$

and so

$$L_h[y_i] = \frac{a(x_i)}{2h} \left[\cot h \left(\frac{a(x_i)h}{2\varepsilon} \right) + 1 \right] y_{i+1} - \frac{a(x_i)}{h} \cot h \left[\frac{a(x_i)h}{2\varepsilon} \right] y_i$$

$$+ \frac{a(x_i)}{2h} \left[\cot h \left(\frac{a(x_i)h}{2\varepsilon} \right) - 1 \right] y_{i-1} = f(x_i) \quad (1.11d)$$

Note that this difference operator is of positive type. Il'in then proves that if a and f are C^2 on the interval, then

$$|y(x_i) - y_i| \leq Kh^2$$

where K is a constant that depends on ε. Kriess and Kriess [87] have discussed some numerical methods for a system of singular perturbation problems of the form

$$\frac{dy}{dx} = A(x)y + F(x), \qquad 0 \leq x \leq 1 \quad (1.12)$$

with n linearly independent boundary conditions

$$R_o y(0) + R_1 y(1) = g \quad (1.13)$$

where $y = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})^T$ is a vector function with n components and R_o, R_1 and $A(x)$ are nxn matrices. Abrahamsson et al. [1] considered schemes applied to singular perturbation problems in which the equation was a system. In the scalar case, their work can be considered as refinement of upstream one-sided difference scheme. The idea is to introduce a parameter into the difference equation and to choose this parameter in such a way that a more accurate approximation for the reduced problem is obtained. They considered the following system of ordinary differential equations

$$\varepsilon y'' + A(x)y' + B(x)y = F(x); \quad 0 \leq x \leq 1 \qquad (1.14)$$

with boundary conditions

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta \qquad (1.15)$$

where $\varepsilon > 0$ is a small parameter, $y = (y^{(1)}, y^{(2)}, \ldots, y^{(n)})^T$, $F = (F^{(1)}, F^{(2)}, \ldots, F^{(n)})^T \in C^\infty$ are vector functions with n components and $A, B \in C^\infty$ are nxn matrices. They assumed that

$$A(x) = \begin{pmatrix} A^I(x) & 0 \\ 0 & A^{II}(x) \end{pmatrix}$$

with

$$A = A^* \text{ (Adj. of A)}, \; A^I \leq -\eta < 0 \; , \; A^{II} \geq \eta > 0$$

for $0 \leq x \leq 1$ and for some constant $\eta > 0$. They devised some schemes which, with net spacing $h >> \varepsilon$, yield 'accurate' solutions in the interior, that is away from the boundary layers.

This is done and Richardson extrapolation is justified under appropriate assumption on $A(x)$ but in general the accuracy can not be better than $O(\varepsilon)$. Specific computations of some interest with difference schemes are contained in Greenspan [61]. Two general purpose codes for two point boundary value problems, that can also be used for singular perturbation problems, have been constructed by Lentini and Pereyara [92] and Ascher et al. [7]. A program that implements an adaptive method is constructed by Bank et al. [17]. From the point of view of finite element method users, the need for spacial ways to treat singular perturbation problems was first recognised by Zienkiewicz et al. [149] where it was stated that a finite element equivalent to upwind differences was needed to avoid the problems of oscillatory solutions obtained with standard numerical approximations. A further algorithm using piecewise linear triangular elements in two-dimensions was proposed by Tabata [136] and a further evaluation of the methods as well as a comparison with equivalent finite difference formulations for the convective-diffusion equation has been given in Zienkiewicz and Heinrich [150]. Some applications of the theory of Babuska [15,16] to the singular perturbation problems have been published by Reinhardt. Major contributions to the applications of finite element methods for singular perturbation problems have come from Mitchell, Zienkiewicz, Babuska, Hemker, Miller, Griffiths, Van Veldhuizen, Reinhardt, and Christie. Co-workers of these mathematicians should be included in the list. Hoppensteadt [68] has discussed

the behavior of the solution of the singular perturbation problems on the infinite interval. They considered the initial value problems of the form

$$\begin{aligned} x' &= f(t,x,y,\varepsilon)\ , \quad x(t_o) = x_o \\ \varepsilon y' &= g(t,x,y,\varepsilon)\ , \quad y(t_o) = y_o \end{aligned} \tag{1.16}$$

where $0 < \varepsilon << 1$ and x, f are real K-dimensional vectors with components $x = (x_1, x_2, \dots, x_K)$ and $f = (f_1, f_2, \dots, f_K)$, respectively and y and g are real J-dimensional vectors with components $y = (y_1, y_2, \dots, y_J)$ and $g = (g_1, g_2, \dots, g_J)$, respectively. The purpose of this paper was to investigate the behavior of solutions of (1.16) as $\varepsilon \to 0$, for $t_o \leq t < \infty$. They continued the discussion in another paper Hoppensteadt [69], where they investigated the properties of solutions of ordinary differential equations with a small parameter. Here, they included the case in which the boundary conditions also depend on ε i.e. $x(t_o) = \alpha(\varepsilon)$ and $y(t_o) = \beta(\varepsilon)$. They also extended these results to the boundary value problems. Harries [64] has described the applicability of differential inequalities in singular perturbation problem by studying the model nonlinear singular perturbation problem

$$\varepsilon y'' + yy' - y = 0\ ; \quad 0 \leq x \leq 1 \tag{1.17}$$

$$\text{with } y(0) = A\ , \text{ and } y(1) = B \tag{1.18}$$

whose solutions exhibit a wide variety of interesting behavior.

Cohen [33] has discussed the existance and asymptotic behavior for small $\varepsilon > 0$ of the solution of the nonlinear two-point boundary value problem

$$\varepsilon y'' + f(x,y,y')y' = 0, \quad 0 < x < 1 \tag{1.19}$$

with

$$y'(0) - ay(0) = A \geq 0 \quad (a > 0) \tag{1.20a}$$

$$y'(1) + by(1) = B > 0 \quad (b > 0) \tag{1.20b}$$

They imposed the following conditions on the nonlinear f in (1.19) :

(i) $f(x,y,y')$ is continuously differentiable in the region

$$R = \{(x,y,y'): 0 \leq x \leq 1,\ 0 \leq y \leq B/b,\ y' \geq 0\}$$

(ii) $y_1 \leq y_2$ and $y_1' \leq y_2'$ imply that

$$f(x,y_1,y_1') \leq f(x,y_2,y_2') \text{ on } R$$

(iii) $f(x,y,y') \geq \beta > 0$ on R

(iv) There exists a constant k such that for all $(x,y,y') \in R$

$$|f(x,y,y') - f(x,z,z')| \leq k(|y-z| + |y'-z'|).$$

Under these conditions, they proved that there exists a solution $y(x,\varepsilon)$ of (1.19), (1.20a-b) such that $y(x,\varepsilon) \to B/b$ and $y'(x,\varepsilon) \to 0$ uniformly on any subinterval $0 < \delta \leq x \leq 1$. The entire analysis is based on the so-called 'shooting method' for ordinary differential equations. Dorr and Parter [40] have discussed the asymptotic behavior as $\varepsilon \to 0$ of solutions

$u(t) = u(t,\varepsilon)$ and $v(t) = v(t,\varepsilon)$ to nonlinear boundary value problems of the form

$$u'' = f(t,u,v) \; ; \; 0 < t < 1, \; u(0) = u(1) = 0 \tag{1.21}$$

$$\begin{aligned} &\varepsilon v'' + g(t,u,u')v' - C(t,u,u')v = 0 \\ &0 < t < 1, \; v(0) = v_o \; ; \; v(1) = v_1 \end{aligned} \tag{1.22}$$

where, they assumed $0 \leq v_o < v_1$ and $C(t,u,u') \geq 0$. They were particularly interested with problems in which there is exactly one interior turning point for equation (1.22) that is, for each $\varepsilon > 0$ there is a unique point $\alpha \in (0,1)$ such that $g(\alpha,u(\alpha),u'(\alpha)) = 0$, and $g(t,u(t),u'(t))$ changes sign in a neighbourhood of $t = \alpha$. In order to study the asymptotic behavior of the solutions they assumed the following conditions :

(i) $0 \leq v_o < v_1$, (ii) $C(t,u,u') \geq 0$

(iii) $f(t,u,u')$, $g(t,u,u')$, and $C(t,u,u')$ are continuous in all variables

(iv) There exists a continuous function $f_o(t,v)$ such that

$$|f(t,u,v)| \leq f_o(t,v) \text{ for } t \in [0,1] \text{ and } v \in [0,v_1].$$

Kopell and Parter [85] have discussed the analysis of the problem

$$\varepsilon y''(t,\varepsilon) = [y^2(t,\varepsilon) - t^2]\, y'(t,\varepsilon) \tag{1.23}$$

with $\quad y(-1,\varepsilon) = A$, and $y(0,\varepsilon) = B \quad$ (1.24)

based entirely on priori estimates and the 'shooting' method.

Depending on the choice of A and B one can **ensure the** existance of turning points. However, due to the nonlinearity of the problem one does not know the position or number of such turning points. The methods of analysis enables one to get a complete picture of the variation in the solutions as A and B are changed. A modified upwind scheme for convective-diffusion equations, which combines the advantages of being stable and of second order is presented by Axelsson and Gustafsson [13]. They considered the convective-diffusion equation in one dimensional case

$$-\mu u'' + vu' = 0 \tag{1.25}$$

$$\text{with} \quad u(0) = \alpha \text{ and } u(1) = \beta \tag{1.26}$$

Finite difference (or finite element) approximations to (1.25) lead to a system of linear equations. In order to obtain a diagonally dominant matrix and preserve order of accuracy $O(h^2)$ in h, they modified the upwind scheme as follows. Since there is no difficulty with the diffusion term ($-\mu u''$) we pay our attention to the convective term (vu'). Clearly

$$u(x_{i+1}) = u(x_i)+(x_{i+1}-x_i)u'(x_i) + \frac{(x_{i+1}-x_i)^2}{2} u''(x_i)+O(h^3)$$

and thus

$$u'(x_i) = \frac{u(x_{i+1})-u(x_i)}{(x_{i+1}-x_i)} - \frac{(x_{i+1}-x_i)}{2} u''(x_i) + O(h^2) \tag{1.27}$$

The first term of the right hand side represents the usual upwind approximation of u'. For $v(x_i) < 0$ we use (1.27) in the approximation of (1.25) and include the term

$$- \frac{(x_{i+1}-x_i)\ v(x_i)\ u''(x_i)}{2}$$

in the diffusion term, which (at the point x_i) will become

$$- [\mu + \frac{(x_{i+1}-x_i)v(x_i)}{2}]\ u''(x_i)$$

$$= -\{\mu/[1 - \frac{(x_{i+1}-x_i)v(x_i)}{2\mu}]\}\ u''(x_i) + O(h^2) \qquad (1.28)$$

Thus the coefficient in front of u'' is always negative so that the matrix will be diagonally dominant. For u'' we use the central difference approximation. For $v(x_i) > 0$ we, instead of (1.27), use

$$u'(x_i) = \frac{u(x_i)-u(x_{i-1})}{(x_i-x_{i-1})} + \frac{(x_i-x_{i-1})}{2}\ u''(x_i) + O(h^2) \qquad (1.29)$$

to obtain the diagonal dominance. Although the approximation (1.28) is bad when $(x_{i+1}-x_i)v(x_i)/(2\mu)$ is large, i.e. v/μ is large, $x_{i+1}-x_i$ does not have to be too small when v/μ is large, the convective term dominates in both the discrete and the continuous model. A class of two point boundary value problems

$$\begin{aligned} x' &= g_1(x,t) + B_1(t)z + C_1(t)u \\ \lambda z' &= g_2(x,t) + B_2(t)z + C_2(t)u \end{aligned} \qquad (1.30)$$

$$\text{with } x(0) = x_o \text{ and } z(0) = z_o \qquad (1.31)$$

where x and z are n and m dimensional state vectors, respectively, u is an r-dimensional control vector, and λ is a nonnegative

scalar parameter and $g_1, g_2, B_1, B_2, C_1, C_2$ are assumed to be infinitely differentiable in all their arguments in an appropriately defined domain, which arise in fixed final time free end point optimal control problems is considered by Sannuti [134]. An asymptotic power series solution of (1.30) is constructed with respect to a parameter whose perturbation changes the differential order of the problem. Kriess and Parter [88] have discussed the behavior of the solutions of the boundary value problems with turning points :

$$\varepsilon y''(x) + f(x,\varepsilon)y'(x) + g(x,\varepsilon)y(x) = 0 \tag{1.32}$$

for $\quad -a \leq x \leq b$ with $y(-a) = A$ and $y(b) = B$ $\qquad$ (1.33)

where $a, b > 0$, $\varepsilon > 0$ and $f(x,\varepsilon)$ has a single simple zero in $[-a, b]$. An extension of Tikhonov's theorem in singular perturbations is studied in Nipp [111]. They considered the autonomous system of ordinary differential equations

$$x' = f(x,y) + \varepsilon f_1(x,y,\varepsilon) \tag{1.34a}$$

$$\varepsilon y' = g(x,y) + \varepsilon g_1(x,y,\varepsilon) \tag{1.34b}$$

together with the initial conditions

$$x(0,\varepsilon) = x^0(\varepsilon) \tag{1.35a}$$

$$y(0,\varepsilon) = y^0(\varepsilon). \tag{1.35b}$$

An asymptotic approximation for solving singular perturbation problems is presented by Finden [53]. The given system

$$\frac{dx}{dt} = f(x,y,t) \quad \text{and} \quad \varepsilon \frac{dy}{dt} = g(x,y,t) \tag{1.36}$$

with the initial conditions

$$x(0) = \alpha \text{ and } y(0) = \beta \tag{1.37}$$

where x, f and α are m-dimensional vectors and, y, g, β are n-dimensional vectors, is replaced by the following system that is not stiff numerically,

$$\begin{aligned} \frac{dx}{dt} &= f(x,y,t) \\ \frac{dy}{dt} &= h(x,y,t,\varepsilon) \end{aligned} \tag{1.38}$$

for some function h. It has been proved that the solution to (1.38) is a second order approximation interms of the small parameter ε. The approximation h for the quantity $g(x,y,t)/\varepsilon$ is given by

$$\frac{g(x,y,t)}{\varepsilon} = h(x,y,t,\varepsilon) + O(\varepsilon^2).$$

Ortiz [119] has discussed the error analysis of Tau method for a class of singular perturbation problems

$$y''(x,\varepsilon) + \frac{P(x)}{\varepsilon}\ y'(x,\varepsilon) = 0;\ 0 \le x \le 1 \tag{1.39}$$

with $\qquad y(0,\varepsilon) = 1 \text{ and } y(1,\varepsilon) = 0 \qquad (1.40)$

where $0 < \varepsilon << 1$ and P(x) is assumed to be a polynomial or polynomial approximation of a function not identically equal to zero and defined in [0,1]. The concept of introducing a parameter into the numerical scheme and then choosing it in order to meet some criterion has been used for Miller [98] for the singular perturbation problem of the form

$$-\varepsilon\, u'' + a_1 u' + a_o u = f \text{ in } \quad \Omega = (0,1) \tag{1.41}$$

$$\text{with } u(0) = u(1) = 0 \tag{1.42}$$

where $a_o \geq 0$ and $a_1 > 0$ are constants. He treats this problem by means of a finite element method. The parameter $\theta = \theta(\varepsilon)$ is chosen so that

$$0 \leq \theta \leq 1, \ \lim_{\varepsilon \to 0} \theta = 0, \ \lim_{\varepsilon \to 0} \frac{a_1 \theta h}{2\varepsilon} = 1 \tag{1.43a}$$

where h is the uniform mesh width. This parameter θ is introduced in the basis $\{\varphi_j\}_1^{N-1}$ as follows :

$$\varphi_j(x) = \begin{cases} (x - x_{j-1})/\theta h & \text{for } x \in [x_{j-1}, x_{j-1}+\theta h] \\ 1 & \text{for } x \in [x_{j-1}+\theta h, x_j] \\ (x_j + \theta h - x)/\theta h & \text{for } x \in [x_j, x_j+\theta h] \\ 0 & \text{otherwise} \end{cases}$$

It is then shown that the finite element scheme is equivalent to

$$\left[-\frac{\varepsilon}{\theta h} + \frac{a_1}{2}\right] u_{j+1} + \left[\frac{2\varepsilon}{\theta h} + a_o h\right] u_j + \left[-\frac{\varepsilon}{\theta h} - \frac{a_1}{2}\right] u_{j-1}$$
$$= h\left[f(x_{j-1}+\theta h) + f(x_j)\right]/2 \tag{1.43b}$$

$j = 1, 2, \ldots, N-1;$ with $u_o = u_N = 0$.

In the limit when $\theta = 1$ (1.43b) is the usual central finite difference scheme for (1.41)

$$-\varepsilon \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 \frac{u_{j+1} - u_{j-1}}{2h} + a_o u_j = f(x_j) \tag{1.43c}$$

for $j = 1, 2, \ldots, N-1;$ with $u_o = u_N = 0$.

On the other hand in the limit when $\varepsilon = 0$ and thus $\theta = 0$, because of (1.43a), (1.43b) becomes the upwind finite difference scheme for the reduced problem of (1.41)

$$a_1 \frac{u_j - u_{j-1}}{h} + a_o u_j = \frac{f(x_{j-1}) + f(x_j)}{2} \tag{1.43d}$$

for $j = 1, 2, \ldots, N-1$; with $u_o = 0$.

A particularly interesting intermediate choice of θ is

$$\theta = [\tanh a_1 h/2\varepsilon]/[a_1 h/2\varepsilon] \tag{1.43e}$$

It is easy to check that the θ defined by (1.43e) satisfies (1.43a), and that for this θ, (1.43b) becomes

$$-\frac{a_1 h}{2} \coth\left(\frac{a_1 h}{2\varepsilon}\right) \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 \frac{u_{j+1} - u_{j-1}}{2h}$$
$$+ a_o u_j = \frac{f(x_{j-1} + \theta h) + f(x_j)}{2} \tag{1.43f}$$

for $j = 1, 2, \ldots, N-1$; with $u_o = u_N = 0$

which is the finite difference scheme introduced by Il'in [76], (apart from the inhomogeneous term which is simply $f(x_i)$), (compare with Eqn. (1.11d)). However, Miller has not carried out any error analysis in this paper for arbitrary θ. The above concept is extended for partial differential equations by Miller [99]. Instead of using the parameter in the elements, Christie et al. [32] places a parameter in the test functions to obtain asymmetric linear and quadratic basis functions for the test space. In this way the oscillations which occur with

symmetric test functions are no longer present. They considered the following problem

$$\varepsilon y'' - Ky' = 0 \qquad 0 < x < 1 \tag{1.44}$$

$$\text{with } u(0) = 1 \text{ and } u(1) = 0 \tag{1.45}$$

They choose $K = 60$. Their scheme yields an upwind differencing via an appropriate choice of the parameter. However, this scheme seems to be of limited value requiring a moderate size of $\varepsilon (= 1/K)$, i.e. $\varepsilon \sim O(10^{-2})$ or $O(10^{-3})$. These authors have continued to investigate this approach and more recent results are contained in Mitchell et al. [104]. Hemker [65] and de Groen and Hemker [63] also considered a modification of the finite element method for problems of the form (1.1). The modification consists of considering test and trial spaces which in addition to the usual piecewise polynomials are supplemented in each subinterval by a piecewise exponential that is a local approximation to the singular solution of the equation $L_\varepsilon[y] = 0$. A so-called 'fitting' function which depends on the coefficient of the first order term is introduced in order that the trial space is fitted exponentially to the singular part of L_ε (i.e. $-\varepsilon \frac{d^2}{dx^2} + a(x) \frac{d}{dx}$). Similarly, the test space is fitted to the singular part of the adjoint equation and gives good approximations to the Green's function of (1.1). Hemker and de Groen showed by proper choice of trial and test spaces the exponentially fitted method applied to

$$-\varepsilon u'' + u' = 0 \qquad 0 < x < 1 \tag{1.46}$$

$$\text{with } u(0) = \alpha \text{ and } u(1) = \beta \tag{1.47}$$

with the fitting function $\alpha(x) = \frac{1-a(x)}{\varepsilon}$ is equivalent to Il'in's scheme. Griffiths [62] discussed some algorithms for convective-diffusion equations of the type

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - K\frac{\partial u}{\partial x}, \; K > 0 \tag{1.48}$$

subject to initial condition and variety of homogeneous boundary conditions. Flaherty and O'Malley [56] have proposed an algorithm for stiff two point boundary value problems of which singularly perturbed problems are an important subclass. They obtained the numerical solution by finding the roots of the characteristic polynomial of the reduced problems. These roots enable them to obtain boundary layer correction terms which are then added to the numerical solutions of the reduced problem to obtain the approximation. Lorenz [95] has considered the numerical solution of singularly perturbed two point boundary value problem

$$-\varepsilon y'' + \alpha(x,y)y' + \beta(x,y) = 0 \tag{1.49}$$

$$\text{for } 0 \le x \le 1 \text{ with } y(0) = A \text{ and } y(1) = B \tag{1.50}$$

He essentially breaks the interval into two parts and uses different meshes on each part to solve a reduced and boundary layer problem. We note that a criterion for choosing where to break the interval is not given although the cutting parameter K

can not be too large. Further, the solution of the reduced problem is used as the data for the boundary layer problem at the cut point. Thus Lorenz solves the reduced problem only upto the cut point and then solves the boundary layer problem for the remainder of the interval. Hoppensteadt and Miranker [70] have discussed an extrapolation method for the numerical solution of singular perturbation problems. They described in detail for initial value problems

$$\frac{dx}{dt} = f(t,x,y,\varepsilon) \; ; \; x(0) = \xi(\varepsilon) \tag{1.51}$$

$$\varepsilon \frac{dy}{dt} = g(t,x,y,\varepsilon) \; ; \; y(0) = \eta(\varepsilon) \tag{1.52}$$

to which the methods of averaging and matched asymptotic expansions can be applied. Fattorini [52] has studied the singular perturbation and boundary layer for an abstract Cauchy problems

$$\varepsilon^2 u''(t,\varepsilon) + u'(t,\varepsilon) = Au(t,\varepsilon) + f(t,\varepsilon) \tag{1.53}$$

for $\quad t \geq 0$ with $u(0,\varepsilon) = u_0(\varepsilon)$ and $u'(0,\varepsilon) = u_1(\varepsilon)$ (1.54)

and $\quad u'(t) = Au(t) + f(t); \; t \geq 0$ with $u(0) = u_0$ (1.55)

This paper also contained a substantial bibliography on singular perturbations of abstract Cauchy problems. Lick and Gaskins [94] have described a consistent and accurate procedure for obtaining difference equations from singularly perturbed differential equations. This procedure is an extension of the integral method and incorporates what is known of the solution of the differential

equation in the formulation of the corresponding difference equation. Although somewhat more complicated to apply than other methods, this procedure has advantages in generality, consistency and accuracy. Veldhuizen [140] has used a finite element method using piecewise polynomials of degree $\leq k$ to approximate the problem

$$\varepsilon u'' + u' = f \; ; \; 0 \leq x \leq 1 \tag{1.56}$$

$$\text{with } u(0) = u(1) = 0 \tag{1.57}$$

with a very irregular mesh h. On this mesh, error estimates of order $O(h^{k+1})$ are obtained uniformly in ε, h the maximum step size, for $k \geq 2$. And, the condition number of the system of linear equations one has to solve in order to get the approximation, is estimated. They extended these results in Veldhuizen [141] and discussed some higher order schemes of positive type for the following class of problems

$$\varepsilon u'' + a(x)u' = f(x) \; ; \; 0 \leq x \leq 1 \tag{1.58}$$

$$\text{with } u(0) = u(1) = 0 \tag{1.59}$$

and $a(x) \geq a_0 > 0$. They also extended these results for elliptic problems of the singular perturbation type in two dimensions. Abrahamsson and Osher [2] have discussed the monotone difference schemes for the problem

$$\varepsilon y'' - [f(y)]' - b(x,y) = 0, \quad 0 \leq x \leq 1 \tag{1.60}$$

$$\text{with } y(0) = A \text{ and } y(1) = B \tag{1.61}$$

where $f \in C^1(R^1)$ and $b \in C^1(R^2)$ and $\frac{\partial b(x,y)}{\partial y} \geq \delta > 0$.
They proved some convergence results and showed that the Engquist-Osher [49] monotone scheme will reproduce essential properties of the true solution for any grid. Osher [120] developed upwind finite difference approximation for nonlinear singular perturbation problems and also for systems of non-linear hyperbolic conservation laws. Similar discussion with one sided difference schemes for the problem

$$\varepsilon y'' - a(y)y' - b(x,y) = F(x) \;,\; -1 \leq x \leq 1 \tag{1.62}$$

$$\text{with } y(-1) = A \text{ and } y(1) = B \tag{1.63}$$

where A, B are constants, $a(y)$, $b(x,y)$ are C^2 functions with $b(x,0) = 0$ and $\frac{\partial b(x,y)}{\partial y} \geq 0$ is contained in Osher [121]. The existence and asymptotic behavior as $\varepsilon \to 0$ of solutions of problems of the following form

$$\varepsilon x'' = f(t,x,x',\varepsilon) \tag{1.64}$$

$$\text{with } x(0,\varepsilon) = A(\varepsilon) \text{ and } x(1,\varepsilon) = B(\varepsilon) \tag{1.65}$$

are studied by Kelley [81], using differential inequality techniques. Conditions are derived under which the problem (1.64) - (1.65) has unique solution which converges uniformly as $\varepsilon \to 0$ outside boundary layers at each end point of width $\sqrt{\varepsilon}$ to a solution of the reduced equation

$$0 = f(t,x,x',0) \tag{1.66}$$

In another paper, Kelley [82] discussed the boundary and interior

layer phenomena for singularly perturbed systems. They established, some sufficient conditions for the existence of solutions of a system

$$\varepsilon Z'' = H(t, Z) \tag{1.67}$$

$$\text{with } Z(0) = A \text{ and } Z(1) = B \tag{1.68}$$

where $Z, H, A,$ and B are vectors, and $0 < \varepsilon << 1$. Kellogg et al. [83] have discussed the analysis of three point difference schemes for singular perturbation problem without turning points. They have obtained bounds for the discretization error which are uniformly valid for all h and $\varepsilon > 0$. The degeneration of the order of the schemes at $\varepsilon = 0$ is also considered. Motivated by the asymptotic behavior of the problems, Sakai [132], and Sakai and Usmani [133] have discussed some numerical methods based on chopping procedures for the solution of two point boundary value problems of singular perturbation type. Berger et al. [23] proved that an exponential tridiagonal difference scheme, when applied with a uniform mesh of size h to :

$$\varepsilon u'' + b(x)u' - d(x)u = f(x);\ 0 \leq x \leq 1 \tag{1.69}$$

$$u(0) = \alpha_0 \text{ and } u(1) = \alpha_1 \tag{1.70}$$

where ε is a small positive parameter; α_0, α_1 are given constants b, d, f are in $C^5[0,1]$; $d(x) \geq 0$ and $b(x) > B_1$ on $[0,1]$ for some positive constant B_1; is uniformly second order accurate (i.e. the maximum of the errors at the grid points is bounded by Ch^2 with the constant independent of h and ε). This scheme was

derived by El-Mistikawy and Werle [46] by a C^1 patching of a pair of piecewise constant coefficient approximate differential equations across a common grid point. Let J be a positive integer and define the uniform mesh length $h = 1/J$. Let the grid points $\{x_i\}$ be given $x_i = ih$, $i = 0,1,\ldots,J$; and let U_i denote the approximate value for $u_i = u(x_i)$. When applied to (1.69), the family of schemes has the form

$$\frac{\varepsilon}{h^2}\left[r_i^- U_{i-1} + r_i^c U_i + r_i^+ U_{i+1}\right] = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1} \qquad (1.71)$$

$$\text{for } i = 1,2,\ldots,J-1; \text{ with } U_0 = \alpha_0 \text{ and } U_J = \alpha_1. \qquad (1.72)$$

The choice of the coefficients r_i^-, r_i^c, r_i^+ and q_i^-, q_i^c, q_i^+ determine the particular scheme. Further details on this type of approach can be found in Berger et al. [23]. A power series expansion in the damping parameter ε of the limit cycle $U(t,\varepsilon)$ of the free Van der Pol equation

$$U'' + \varepsilon(U^2-1)\,U' + U = 0 \qquad (1.73)$$

is constructed and analyzed by Dadfar et al. [36]. Grasman and Matkousky [60] have employed a variational formulation of the problem (1.32) with (1.33) to resolve the question of the number and location of the boundary layers as well as to uniquely determine the asymptotic expansion of the solution. These results are then extended to analogous problems for partial differential equations with turning points. Similar results are discussed by Grasman [59] for a class of elliptic singular perturbation problems. Consideration of the exact

solution of singularly perturbed equations provides insight into the appropriate use of perturbation techniques. Such a study is made by Lange and Miura [89] for a class of linear singular perturbation problems. Collocation methods using both cubic polynomials and splines in tension are developed by Flaherty and Mathon [55] for the following class of problems

$$Ly = \varepsilon y'' + p(x)y' + q(x)y = f(x) \; ; \; a \leq x \leq b \tag{1.74}$$

with $\alpha_{11}y(a,\varepsilon) + \alpha_{12}y'(a,\varepsilon) = A$ and $\alpha_{21}y(b,\varepsilon) + \alpha_{22}y'(b,\varepsilon) = B$ (1.75)

Niijima [108] derived some sufficient conditions for the problem

$$\varepsilon y'' + f(x,\varepsilon)y' + g(x,\varepsilon)y = h(x,\varepsilon); \; -a \leq x \leq b \tag{1.76}$$

$$\text{with } y(-a) = A(\varepsilon) \text{ and } y(b) = B(\varepsilon) \tag{1.77}$$

to have a unique solution. They presented, in Niijima [109], a difference scheme for a semilinear problem with any number of turning points of arbitrary orders. The class of the problems considered are

$$\varepsilon y'' - [a(x)y]' - b(x,y) = 0; \quad 0 \leq x \leq 1 \tag{1.78}$$

$$\text{with } y(0) = A \text{ and } y(1) = B \tag{1.79}$$

They showed that a solution of their scheme converges uniformly in ε, to that of continuous problem. A completely exponentially fitted difference scheme is derived for turning point problems

by Niijima [110]. An error analysis of that scheme is also given under some conditions. Howes [73] presented some results and sufficient conditions in order that the solution of the problem (scalar and vector form)

$$\varepsilon y'' = F(y)y' + g(x,y), \quad a \leq x \leq b \tag{1.80}$$

$$\text{with } y(a,\varepsilon) = \alpha \text{ and } y(b,\varepsilon) = \beta \;, \tag{1.81}$$

display boundary layer behavior at an endpoint. They also discussed the boundary layer behavior of singularly perturbed initial value problems (in Howes [71]) and also for the third order boundary value problems (in Howes [72]). Weiss [146] derived the stability and convergence results for the box and trapezoidal schemes applied to singularly perturbed boundary value problems. Lorenz [96] also discussed the analysis of the problem (1.80) - (1.81) but with scalar case only. Reinhardt [125] has discussed the stability of some difference scheme applied to the problem (1.1).They considered the difference approximation to (1.1) as

$$B_{k,\varepsilon}\, y_{k+1} + A_{k,\varepsilon}\, y_k + \frac{\varepsilon}{h}\, y_{k-1} = h \quad f(x_k); \; k = 0,\ldots,N-1 \tag{1.82a}$$

$$\text{with } y_o = y(0) = \alpha \text{ and } y_N = y(1) = \beta \tag{1.82b}$$

where

$$A_{k,\varepsilon} = -(a(x_k) - hb(x_k) + 2\,\frac{\varepsilon}{h}) \tag{1.82c}$$

$$B_{k,\varepsilon} = a(x_k) + \frac{\varepsilon}{h} \tag{1.82d}$$

In another paper, Reinhardt [126] has discussed the posteriori error analysis and adaptic finite element methods for solving

singularly perturbed convective-diffusion equations. O'Malley and Flaherty [116] have discussed some analytical and numerical methods for nonlinear singular-singularly perturbed initial value problems. Ascher [6] has discussed some difference scheme for singular-singularly perturbed boundary value problems. These results are the extensions of Ascher and Weiss [8,9,10]. Singular-singularly perturbed boundary value problems are also considered by O'Malley [114]. Recently, Roberts [128] has given a boundary value technique to solve singular perturbation problems. It is based on the shooting methods. He also discussed (in Roberts [129]), the analytical and approximate solutions of the problem :

$$\varepsilon y'' = yy' , \quad -1 \le x \le 1 \tag{1.83}$$

$$y(-1) = \alpha \text{ and } y(1) = \beta \tag{1.84}$$

More recently, Roberts [130] has extended his boundary value technique to solve the problem :

$$\varepsilon y'' + yy' - y = 0 ; \quad 0 \le x \le 1 \tag{1.85}$$

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta \tag{1.86}$$

Very recently O'Riordan [118] introduced a new piecewise linear finite element, which is designed to handle singular perturbation problems. They first introduced the new concept of hinged elements. Then he examined finite element approximations, to the problems of the form

$$\varepsilon y'' + ay' = f ; \qquad 0 \le x \le 1 \tag{1.87}$$

$$\text{with } y(0) = y(1) = 0 \tag{1.88}$$

whose errors are uniformly in ε good, i.e. the error constants and asymptotic order of convergence as $h \to 0$ are independent of ε and the mesh size h. Piecewise exponential elements yield

$$\max_{0 \leq i \leq N} |y(x_i) - y^h(x_i)| \leq Ch^2 \text{ uniformly in } \varepsilon$$

$$||y-y^h||_\infty \leq Ch \text{ uniformly in } \varepsilon.$$

He introduced 'hinged functions' (piecewise linear elements), which retain

$$\max_{0 \leq i \leq N} |y(x_i) - y^h(x_i)| \leq Ch^2 \text{ uniformly in } \varepsilon$$

but globally are not uniform (w.r.t. ε) O(h) (are $O(h^{1/2})$ uniformly) in the L^2 norm. However, when either $h \ll \varepsilon$ or $\varepsilon \ll h$, then

$$||y-y^h||_2 \leq Ch \text{ where C is independent of } \varepsilon \text{ and h.}$$

Similar results are generalized to two-dimensions. Further details may be found in O'Riordan [117]. Hsiao and Jordan [74] have discussed numerical schemes based on the method of matched asymptotic expansions and modifying the boundary layer problem. The particular numerical method they used for solving the modified problems in Galerkin method with linear finite elements as trial functions. They have applied these schemes successfully to several examples. Finally, an error analysis has been performed and encompasses both numerical and singular perturbation theory. A detailed study of this approach may be found in Jordan [78].

Recently, Brandt [25] presented the multi level adaptive technique for solving general singular perturbation problems. Further discussion on the similar results is contained in Brandt [26]. Ringhofer [127] has presented a numerical method based on collocation with polynomial splines for quasi-linear systems of singularly perturbed boundary value problems. The stability properties of the associated difference operator are examined and a variable mesh algorithm to achieve a certain over-all accuracy is developed. The number of grid points required by the algorithm is also estimated. A detailed analysis is performed for a finite element method applied to the general one dimensional convective diffusion problem by Szymczak et al. [135]. Jain et al. [77] derived a third order variable mesh difference method for the numerical solution of two point, second order, singular perturbation problems. Herbest et al. [67] presented a generalized Petrov-Galerkin method for the numerical solution of Burger's equation.

Fitzsimons et al. [54] have introduced Petrov-Galerkin finite element methods with a hinged test space for singularly perturbed, second order, ordinary, linear differential equations. They discussed Petrov-Galerkin methods on a uniform mesh which have asymptotic error estimates, as the mesh size tends to zero, whose magnitude is independent of the singular perturbation parameter. This is a marked contrast to standard finite element methods which donot possess such a property on a uniform mesh. For these, the error on a fixed uniform mesh

blows up as the singular perturbation parameter tends to zero. This robust behaviour of these Petrov-Galerkin methods for singular perturbation problems is achieved by choosing trial spaces of standard piecewise polynomial type, while the test spaces consist of hinged piecewise polynomials. They also described a number of sample problems and presented the numerical results which are found to be in good agreement with those expected from the theoretical considerations.

Very recently, Axelsson and Carey [11] considered a class of singularly perturbed two point boundary value problems with various types of boundary conditions and examined the layer behaviour of the solution. By constructing a more regular modified problem with a correction term they discussed how to deal with the boundary layer separately and proved error estimates which are uniform in the singular perturbation parameter.

1.4 CONCLUSIONS AND FURTHER DEVELOPMENTS

Throughout the preceeding review, we have mentioned various topics in need of further study. One of the most important goals of course is the determination of which are the most accurate and efficient codes for solving general singular perturbation problems. It is clear that the competition is between asymptotic methods and numerical methods. The problems like, singular-singular perturbation problems, singular perturbation problem with turning point(s), nonlinear singular perturbation problems, elliptic/parabolic/hyperbolic singular

perturbations, general singularly perturbed partial differential equations are now active research problems in the area of singular perturbations. The general motivation of the numerical analysts is to provide more efficient, simpler, powerful, computational techniques to solve singular perturbation problems. Also, we are particularly interested in the current activity in this field. Finally, we note that most of the papers in our survey have contained some computed results. This is a very healthy sign and of course it demonstrates the efficiency and applicability of the method.

1.5 SUMMARY OF THE THESIS

As we have seen in Section 1.2, there are a wide variety of asymptotic expansion methods for solving singular perturbation problems. But there can be difficulties in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions which are not routine exercises but require skill, insight, and experimentation. Even the matching of the coefficients of the inner and outer solution expansions can be a demanding process. In view of the wealth of literature on singular perturbation problems, and in view of the specialized skills and experience that experts in the field deem necessary, we raise the question whether there may be other ways to attack singular perturbation problems; ways that are very easy to use, and ready for computer implementation; ways that are more

accessible to the practicing engineer or applied mathematician. An engineer or applied mathematician needs efficient techniques which he can apply routinely in order to avoid dealing with methods that may require a great deal of experimentation and which may or may not yield an answer.

In response to this need for a fresh approach to singular perturbation problems, we have proposed and illustrated in this thesis some efficient and simple numerical techniques which can be applied routinely to solve singularly perturbed two point boundary value problems in ordinary differential equations.

In Chapter 2, an approximate method for the numerical integration of a class of linear singularly perturbed two point boundary value problems with a boundary layer on the left end of the underlying interval is presented. This method does not depend on asymptotic expansions. The original second order differential equation is replaced by an approximate first order differential equation of neutral type with a small deviating argument. Then, the trapezoidal formula is used to obtain the three-term recurrence relationship. Discrete invariant imbedding algorithm is used to solve the tridiagonal algebraic system. The stability of this algorithm is investigated. The proposed method is iterative on the deviating argument. Some numerical examples have been solved to demonstrate the applicability of the method.

A nonasymptotic method for solving linear singularly perturbed two point boundary value problems with a boundary layer

on the left end of the interval is presented in Chapter 3. This is an alternative method to that of Chapter 2, and also simpler than that method. The method is distinguished by the following fact : The original second order differential equation is replaced by an approximate first order differential equation with a small deviating argument, and solved efficiently by employing the Simpson's rule coupled with discrete invariant imbedding algorithm. The proposed method is iterative on the deviating argument. Some numerical experiments have been included to demonstrate the efficiency of the method.

In order to know the behavior of the solution of a singular perturbation problem it is always suggestive to divide the problem into two problems and to solve them separately. Keeping this in mind, a new approach based on the method of inner boundary condition is presented in Chapter 4. The method is distinguished by the following facts : The original problem is partitioned into inner and outer region differential equation systems. Asymptotic expansion is used to obtain the terminal boundary condition. Using an appropriate transformation, a new inner region problem is obtained and solved as a two point boundary value problem. The derivative boundary condition at the terminal point is then derived from the solution of the inner region problem. Using this condition, the outer region problem is efficiently solved by employing the classical second order central finite difference scheme. The proposed method is iterative on the terminal point of the inner region. Some test

examples have been solved to illustrate the method and the computational results are compared with exact solutions.

In Chapter 5, a boundary layer technique is presented for a class of linear singular perturbation problems. It is motivated by the asymptotic behavior of the singular perturbation problem. As in Chapter 4, the original problem is divided into inner and outer region problems. However, asymptotic expansions are not employed. The reduced problem is solved to obtain the terminal boundary condition. Then, a new inner region problem is created and solved as a two point boundary value problem. In turn, the outer region problem is also modified and the resulting problem is efficiently solved by employing the trapezoidal formula coupled with discrete invariant imbedding algorithm. This technique is also iterative on the terminal point of the inner region. Numerical experience with three examples is reported.

In Chapter 6, two terminal boundary value techniques are presented for linear singular perturbation problems. These involve a modification of the singular perturbation problem so that any suitable existing standard numerical method can be employed on the modified problem. As usual, the original problem is divided into outer and inner region problems. Two techniques are introduced to obtain the terminal boundary condition in the implicit form. Then, the outer region problem is solved as a two point boundary value problem and an explicit terminal boundary

condition is obtained. In turn, the inner region problem is modified and solved as a two point boundary value problem using the explicit terminal boundary condition. Three numerical examples are included to illustrate these techniques.

An approximate method for solving a class of singular perturbation problems is presented in Chapter 7. It is designed on the basis of the asymptotic behavior of the singular perturbation problem. As with other methods, the given region is divided into inner and outer regions. The original second order problem is replaced by an asymptotically equivalent first order problem and solved as an initial value problem in the inner region. A terminal boundary condition is then obtained from the solution of the inner region problem. In turn, an outer region problem is obtained, by replacing the second order differential equation by an approximate first order differential equation with a small deviating argument, and solved efficiently by employing the trapezoidal formula coupled with discrete invariant imbedding algorithm. Several numerical examples have been solved to demonstrate the applicability of the method. Finally the method is extended to a more general class of problems. Again one numerical example is solved in this general class.

In Chapter 8, an initial value technique, which is simple to use and easy to implement, is presented for a class of nonlinear singular perturbation problems with a boundary layer on the left end of the interval. It is distinguished by the following fact : The original second order problem is replaced by

an asymptotically equivalent first order problem and solved as an initial value problem. Numerical experience with several examples is described.

A boundary value method for a class of nonlinear singular perturbation problems is presented in Chapter 9. By constructing a modified problem with a boundary layer correction, a discussion on how to deal with the boundary layer separately is presented. The proposed method is iterative on the terminal point of the boundary layer region. Several numerical examples are discussed to illustrate the method. Finally, a lower bound for the terminal point of the boundary layer region is obtained in terms of the perturbation parameter.

Finally in Chapter 10, the nonasymptotic method developed in Chapter 3 has been extended for solving general linear singularly perturbed two point boundary value problems. Firstly, problems with a right end boundary layer have been discussed. Secondly, problems with an interior layer have been discussed. Finally, problems with two boundary layers have been discussed. Numerical experience with the method for some model problems is also reported to confirm the theoretical analysis.

In a nut-shell, the numerical methods presented in this thesis for solving singularly perturbed two point boundary value problems in ordinary differential equations have been shown to be efficient over the conventional methods. Above all, these methods are conceptually simple, easy to use, and are

readily adapted for computer implementation with a modest amount of problem preparation. Several model linear and non-linear problems have been solved and numerical results are presented in respective chapters. It is observed that the accuracy predicted can always be achieved with very little computational effort. All the numerical results, presented in this thesis, have been computed on DEC-1090 computer system at IIT Kanpur.

CHAPTER 2

NUMERICAL INTEGRATION OF A CLASS OF SINGULAR PERTURBATION PROBLEMS

2.1 INTRODUCTION

The numerical treatment of singular perturbation problems has always been far from trivial, because of the boundary layer behavior of the solutions. The well known method of Matched Asymptotic Expansions, which consists of (a) dividing the problem into an inner region problem and an outer region problem, (b) expressing the inner and outer solutions as asymptotic expansions, (c) equating various terms in the inner and outer solutions to determine the constants, and (d) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution, is not a routine exercise but requires skill and experimentation effort. The classical finite difference scheme requires the use of a very fine mesh in the neighbourhood of the boundary layer, so that typical features of the boundary layer will not be lost. This procedure has been successfully applied to several problems by Pearson [122] and many others. However, the disadvantage of this approach is that it requires considerable computational effort. In fact, often because of the limitation of the computer system, the discrete problems involved may become ill-posed numerically when mesh sizes get too small.

The object of this chapter is to present an approximate method for the numerical integration of a class of linear singularly perturbed two point boundary value problems in ordinary differential equations with a boundary layer on the left end of the underlying interval. This method does not depend on asymptotic expansions. The main feature of this method is that it does not require very fine mesh size. We replace the original singular perturbation problem by an approximate first order differential equation of neutral type with a small deviating argument. This replacement is significant from the computational point of view. Theory and discussion on the differential equations with a deviating argument can be found in Elsgolts and Norkin [48], Elsgolts [47], Driver [42], Bellman and Cooke [20], Bellman et al. [19], and Norkin [112]. We use the trapezoidal formula for the numerical integration of the first order differential equation of neutral type with a small deviating argument to obtain the three-term recurrence relationship. Tridiagonal algebraic system is efficiently solved by employing a discrete invariant imbedding algorithm. The stability of this algorithm is investigated. The proposed method is to be repeated for different choices of the deviating argument, until the solution profile stabilizes. Some numerical experiments have been included to demonstrate the applicability of the method.

2.2 NUMERICAL SCHEME

To fix the ideas, we consider the following Singular Perturbation Problem (SPP);

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x); 0 \leq x \leq 1 \qquad (2.1)$$

$$y(0) = \alpha \text{ and } y(1) = \beta, \qquad (2.2)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$ and $b(x) \geq 0$, $a(x) \geq M > 0$ on $[0,1]$ where M is some positive constant. Under these assumptions, (2.1) - (2.2) has a unique solution $y(x)$ which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

Let δ be a small positive deviating argument $(0 < \delta << 1)$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \qquad (2.3)$$

and, consequently, the equation (2.1) is replaced by the following first order differential equation with a small deviating argument :

$$\varepsilon y'(x) - \varepsilon y'(x-\delta) + \delta a(x)y'(x) - \delta b(x)y(x) = \delta f(x) \qquad (2.4)$$

for $\delta \leq x \leq 1$. Transition from the equation (2.1) to equation (2.4) is admitted, because of the condition that δ is small $(0 < \delta << 1)$. Further details on the validity of this

transition can be found in Elsgolts and Norkin [48], Pages : 243 and 244. Rearrangement of the equation (2.4) is sometimes called the differential equation of 'neutral type' with a small deviating argument, namely

$$y'(x) = p(x)y'(x-\delta)+q(x)y(x)+r(x) \tag{2.5}$$

for $\delta \leq x \leq 1$ where

$$p(x) = \frac{\varepsilon}{\varepsilon + \delta a(x)} \tag{2.6}$$

$$q(x) = \frac{\delta b(x)}{\varepsilon + \delta a(x)} \tag{2.7}$$

$$r(x) = \frac{\delta f(x)}{\varepsilon + \delta a(x)} . \tag{2.8}$$

We now divide the interval $[0,1]$ into N equal parts with mesh size h,

$$\text{i.e. } h = \frac{1}{N} \text{ and } x_i = ih \quad \text{for } i = 0,1,2,\ldots,N.$$

Integrating by parts the equation (2.5) in $[x_i, x_{i+1}]$, $(i = 1,2,\ldots,N-1)$, we get

$$\begin{aligned} y(x_{i+1})-y(x_i) &= \int_{x_i}^{x_{i+1}} [\, p(x)y'(x-\delta)+q(x)y(x)+r(x)] \, dx \\ &= p(x_{i+1})y(x_{i+1}-\delta)-p(x_i)y(x_i-\delta) \\ &\quad + \int_{x_i}^{x_{i+1}} [\, -p'(x)y(x-\delta)+q(x)y(x)+r(x)] \, dx. \end{aligned}$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$y(x_{i+1})-y(x_i) = [p(x_{i+1}) - \frac{h}{2} p'(x_{i+1})]\, y(x_{i+1}-\delta)$$
$$- [p(x_i) + \frac{h}{2} p'(x_i)]\, y(x_i-\delta)$$
$$+ \frac{h}{2} [q(x_{i+1})y(x_{i+1})+q(x_i)y(x_i)]$$
$$+ \frac{h}{2} [r(x_{i+1})+r(x_i)] \; . \tag{2.9}$$

Again, by means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) \tag{2.10}$$

and, then by approximating $y'(x)$ by linear interpolation, we get

$$y(x_i-\delta) \approx y(x_i) - \delta \left(\frac{y(x_i) - y(x_{i-1})}{h}\right)$$
$$= (1 - \frac{\delta}{h}) y(x_i) + \frac{\delta}{h}\, y(x_{i-1}), \tag{2.11a}$$

and

$$y(x_{i+1}-\delta) \approx (1 - \frac{\delta}{h}) y(x_{i+1}) + \frac{\delta}{h}\, y(x_i). \tag{2.11b}$$

Hence, by making use of (2.11a-b) in (2.9) leads after simple manipulation to the final three-term recurrence relationship, namely

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{2.12}$$

for $i = 1, 2, \ldots, N-1$; where

$$E_i = \frac{\delta}{h} [p_i + \frac{h}{2} p_i'] \tag{2.13}$$

$$F_i = 1 + \frac{\delta}{h} [p_{i+1} - \frac{h}{2} p_{i+1}'] - (1 - \frac{\delta}{h}) [p_i + \frac{h}{2} p_i'] + \frac{h}{2} q_i \tag{2.14}$$

$$G_i = 1 - (1 - \frac{\delta}{h}) [p_{i+1} - \frac{h}{2} p_{i+1}'] - \frac{h}{2} q_{i+1} \tag{2.15}$$

$$H_i = \frac{h}{2} [r_{i+1} + r_i] \; . \tag{2.16}$$

any $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Equation (2.12) gives a system of (N−1) equations with (N+1) unknowns y_o to y_N. The two given boundary conditions (2.2) together with (N−1) equations are then sufficient to solve for the unknowns y_i's. The matrix problem associated with (2.12) is tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient and stable method called 'Discrete Invariant Imbedding'.

2.3 DISCRETE INVARIANT IMBEDDING

In order to solve the tridiagonal system

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{2.17}$$

for $i = 1, 2, \ldots, N-1$, where E_i, F_i, G_i and H_i are given by the equations (2.13) − (2.16) respectively, subject to the boundary conditions

$$y_o = y(0) = \alpha \tag{2.18a}$$

$$y_N = y(1) = \beta, \tag{2.18b}$$

we make use of the method of discrete invariant imbedding (Angel and Bellman [4]). We seek a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \tag{2.19}$$

where W_i and T_i corresponding to $W(x_i)$ and $T(x_i)$ are to be determined. From (2.19) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \, . \tag{2.20a}$$

Substituting (2.20a) in (2.17), we get

$$E_i(W_{i-1}y_i+T_{i-1}) - F_iy_i + G_iy_{i+1} = H_i$$

$$y_i = \frac{G_i}{F_i-E_iW_{i-1}} y_{i+1} + \frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}} . \tag{2.20b}$$

By comparing (2.20b) and (2.19), we get

$$W_i = \frac{G_i}{F_i-E_iW_{i-1}} , \tag{2.21a}$$

and

$$T_i = \frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}} . \tag{2.21b}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$; we need to know the initial conditions for W_o and T_o. This can be done by considering (2.18a)

$$y_o = \alpha = W_oy_1 + T_o. \tag{2.21c}$$

If we choose $W_o = 0$, then $T_o = \alpha$. With these initial values, we compute sequentially W_i and T_i for $i = 1, 2, \ldots, N-1$; from (2.21a) and (2.21b) in the forward process and then obtain y_i in the backward process from (2.19) using (2.18b).

2.4 STABILITY

We will now show that the discrete invariant imbedding algorithm is computationally stable. By stability, we mean the effect of error made in one stage of calculation is not propagated into larger errors at later stages of calculations. In other words, the local errors are not magnified by further computation.

Let us examine the recurrence relation given by (2.21a). Suppose a small error e_{i-1} has been made in the calculation of W_{i-1}, then we have

$$\bar{W}_{i-1} = W_{i-1} + e_{i-1} .$$

and we are actually calculating

$$\bar{W}_i = \frac{G_i}{F_i - E_i \bar{W}_{i-1}} . \tag{2.22}$$

From (2.22) and (2.21a), we have

$$e_i = \frac{G_i}{F_i - E_i(W_{i-1}+e_{i-1})} - \frac{G_i}{F_i - E_i W_{i-1}}$$

$$= \frac{G_i \; E_i \; e_{i-1}}{[F_i - E_i(W_{i-1}+e_{i-1})] \; [F_i - E_i W_{i-1}]}$$

$$e_i = [\frac{W_i^2 \; E_i}{G_i}] \; e_{i-1} \tag{2.23}$$

under the assumption that initially the error is small. Let us assume that $E_i > 0$ and $G_i > 0$ for $i = 1, 2, \ldots, N-1$; then from the assumptions made earlier that $a(x) > 0$ and $b(x) \geq 0$, we have

$$F_i \geq E_i + G_i \qquad \text{for } i = 1, 2, \ldots, N-1; \tag{2.24a}$$

provided $[q_{i+1}+q_i] \geq [p'_{i+1}+p'_i] - \frac{2}{h}[p_{i+1}-p_i]$.

From the initial condition of W_o, it is clear that $|W_o| < 1$.

From (2.21a)

$$W_1 = \frac{G_1}{F_1} < 1 \quad \text{since } F_1 > G_1$$

$$W_2 = \frac{G_2}{F_2 - E_2 W_1} < \frac{G_2}{F_2 - E_2} \quad \text{since } W_1 < 1$$

$$< \frac{G_2}{E_2 + G_2 - E_2} = 1 \quad \text{since } F_2 \geq E_2 + G_2$$

successively it follows that

$$|W_i| < 1 \text{ for } i = 1, 2, \ldots, N-1. \qquad (2.24b)$$

Then it follows from the equation (2.23) that

$$|e_i| = |W_i|^2 \left|\frac{E_i}{G_i}\right| |e_{i-1}|$$

$$< |e_{i-1}| \text{ provided } |E_i| \leq |G_i| \qquad (2.25)$$

and thus the recurrence relation (2.21a) is stable. Let us now examine the recurrence relation given by (2.21b). Suppose a small error c_{i-1} has been made in the calculation of T_{i-1}, then we have

$$\bar{T}_{i-1} = T_{i-1} + c_{i-1}$$

and we are actually calculating

$$\bar{T}_i = \frac{E_i \bar{T}_{i-1} - H_i}{F_i - E_i W_{i-1}} \; . \qquad (2.26)$$

From (2.26) and (2.21b), we have

$$c_i = \frac{E_i(T_{i-1} + c_{i-1}) - H_i}{F_i - E_i W_{i-1}} - \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}$$

$$= \frac{E_i T_{i-1} + E_i c_{i-1} - H_i - E_i T_{i-1} + H_i}{F_i - E_i W_{i-1}}$$

2.5 NUMERICAL EXPERIMENTS

In this section, we present three numerical experiments to demonstrate the applicability of the method described in the previous sections.

Example 2.1 : Consider the following homogeneous SPP from Bender and Orszag [22], Page : 480; Problem : 9.17 with $\alpha = 0$;

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad 0 \le x \le 1 \tag{2.30a}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1. \tag{2.30b}$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{(e^{m_2} - e^{m_1})}$$

where $m_1 = \dfrac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \dfrac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$.

The computational results are presented in the Table 2.1, 2.2, for $\varepsilon = 10^{-3}$, 10^{-4}, respectively.

Example 2.2 : Now consider the non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [125], Example : 2;

$$\varepsilon y''(x) + y'(x) = 1+2x, \quad 0 \le x \le 1 \tag{2.31a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{2.31b}$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon-1)\,\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}.$$

The computational results are presented in the Table 2.3, 2.4, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 2.3 : Finally, we consider the following SPP with variable coefficients from Kevorkian and Cole [84], Page : 33; Equations : 2.3.26 and 2.3.27 with $\alpha = \frac{-1}{2}$;

$$\varepsilon y''(x) + (1 - \tfrac{x}{2})y'(x) - \tfrac{1}{2}\, y(x) = 0, \; 0 \leq x \leq 1 \qquad (2.32a)$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \qquad (2.32b)$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [106], Page : 148; Equation : 4.2.32) as our 'exact' solution,

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp\left(-(x - \frac{x^2}{4})/\varepsilon\right).$$

The computational results are presented in the Table 2.5, 2.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

2.6 DISCUSSION AND CONCLUSIONS

We have described an approximate method for the numerical integration of a class of linear singularly perturbed two point boundary value problems. This is a practical method and can be easily implemented on a computer to solve such problems. As mentioned, the method is iterative on the deviating argument δ. The scheme is to be repeated for various choices of δ (deviating argument), until the solution profiles stabilize. The choice of δ is not unique but can assume several values satisfying

the condition, $0 < \delta << 1$. To reduce the amount of computation, we fix the mesh size h and vary the deviating argument δ. Finally, we pick up the smallest value of δ which produces the required accuracy. The main feature of the present method is that it does not require very fine mesh size. We have implemented this method on three examples, a homogeneous SPP, a non-homogeneous SPP and a SPP with variable coefficients, by taking different values for ε. Computational results are presented in the Tables 2.1 - 2.6. We have given here only a few values although the solutions are computed at all the points with mesh size h. It can be observed from the tables that the present method approximates the exact solutions very well.

Table 2.1

Computational results for Example 2.1, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	10ε y(x)	1ε y(x)	0.1ε y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.02	0.38083740	0.38083821	0.38084013	0.37567774
0.04	0.38329647	0.38329729	0.38329923	0.38325930
0.06	0.39098915	0.39098998	0.39099191	0.39099386
0.08	0.39887946	0.39888030	0.39888221	0.39888451
0.1	0.40692936	0.40693019	0.40693211	0.40693440
0.2	0.44968231	0.44968313	0.44968501	0.44968726
0.3	0.49692697	0.49692777	0.49692959	0.49693177
0.4	0.54913528	0.54913604	0.54913776	0.54913982
0.5	0.60682873	0.60682941	0.60683101	0.60683289
0.6	0.67058358	0.67058417	0.67058557	0.67058726
0.7	0.74103664	0.74103715	0.74103830	0.74103969
0.8	0.81889167	0.81889205	0.81889290	0.81889392
0.9	0.90492634	0.90492653	0.90492702	0.90492758
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 2.2

Computational results for Example 2.1, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	10ε y(x)	1ε y(x)	0.1ε y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.02	0.37540585	0.37540724	0.37540421	0.37533228
0.04	0.38292628	0.38292767	0.38292464	0.38291405
0.06	0.39066119	0.39066258	0.39065955	0.39064898
0.08	0.39855235	0.39855373	0.39855070	0.39854015
0.1	0.40660290	0.40660428	0.40660126	0.40659073
0.2	0.44936161	0.44936298	0.44936001	0.44934966
0.3	0.49661688	0.49661820	0.49661532	0.49660532
0.4	0.54884154	0.54884279	0.54884007	0.54883059
0.5	0.60655821	0.60655935	0.60655684	0.60654812
0.6	0.67034440	0.67034541	0.67034319	0.67033548
0.7	0.74083842	0.74083924	0.74083740	0.74083101
0.8	0.81874564	0.81874624	0.81874489	0.81874018
0.9	0.90484565	0.90484597	0.90484523	0.90484262
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 2.3

Computational results for Example 2.2, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	0.1ε y(x)	1ε y(x)	10ε y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.96939247	-0.96939178	-0.96939189	-0.97764000
0.04	-0.95641220	-0.95641156	-0.95641167	-0.95648000
0.06	-0.93451978	-0.93451919	-0.93451929	-0.93452001
0.08	-0.91176031	-0.91175979	-0.91175987	-0.91176000
0.1	-0.88820028	-0.88819983	-0.88820001	-0.88820001
0.2	-0.75840011	-0.75840001	-0.75840002	-0.75840001
0.3	-0.60860002	-0.60860015	-0.60860001	-0.60860001
0.4	-0.43879992	-0.43880029	-0.43880001	-0.43880001
0.5	-0.24899985	-0.24900036	-0.24900000	-0.24900001
0.6	-0.03919980	-0.03920038	-0.03919999	-0.03920000
0.7	0.19060021	0.19059963	0.19060001	0.19060000
0.8	0.44040019	0.44039970	0.44040000	0.44039998
0.9	0.71020012	0.71019983	0.71020000	0.71019998
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 2.4

Computational results for Example 2.2, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	0.1ε y(x)	1ε y(x)	10ε y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.02	-0.97930482	-0.97930621	-0.97930598	-0.97940400
0.04	-0.95820699	-0.95820819	-0.95820799	-0.95820799
0.06	-0.93621116	-0.93621216	-0.93621200	-0.93621200
0.08	-0.91341533	-0.91341613	-0.91341601	-0.91341600
0.1	-0.88981949	-0.88982010	-0.88982000	-0.88982000
0.2	-0.75984022	-0.75983995	-0.75984000	-0.75984000
0.3	-0.60986083	-0.60985987	-0.60986000	-0.60986000
0.4	-0.43988129	-0.43987979	-0.43988000	-0.43988000
0.5	-0.24990158	-0.24989973	-0.24990000	-0.24990000
0.6	-0.03992169	-0.03991970	-0.03991998	-0.03991999
0.7	0.19005839	0.19006029	0.19006002	0.19006000
0.8	0.44003868	0.44004023	0.44004002	0.44003999
0.9	0.71001922	0.71002015	0.71002001	0.71002001
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 2.5

Computational results for Example 2.3, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	0.1ε y(x)	1ε y(x)	10ε y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.50159473	0.50159419	0.50158872	0.50505050
0.04	0.51091010	0.51090958	0.51090406	0.51020408
0.06	0.51620755	0.51620705	0.51620152	0.51546392
0.08	0.52157918	0.52157880	0.52157324	0.52083334
0.1	0.52706355	0.52706312	0.52705756	0.52631579
0.2	0.55631049	0.55630967	0.55630407	0.55555556
0.3	0.58899110	0.58899026	0.58898475	0.58823530
0.4	0.62574788	0.62574714	0.62574168	0.62500000
0.5	0.66739370	0.66739284	0.66738757	0.66666666
0.6	0.71497241	0.71497130	0.71496628	0.71428571
0.7	0.76984640	0.76984574	0.76984129	0.76923077
0.8	0.83383209	0.83383141	0.83382790	0.83333333
0.9	0.90939926	0.90939917	0.90939704	0.90909090
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 2.6

Computational results for Example 2.3, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	0.1ε y(x)	1ε y(x)	10ε y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.02	0.50507540	0.50507437	0.50506902	0.50505050
0.04	0.51027886	0.51027793	0.51027257	0.51020408
0.06	0.51553895	0.51553801	0.51553260	0.51546392
0.08	0.52090857	0.52090763	0.52090220	0.52083334
0.1	0.52639124	0.52639028	0.52638485	0.52631579
0.2	0.55563182	0.55563077	0.55562529	0.55555556
0.3	0.58831143	0.58831061	0.58830518	0.58823530
0.4	0.62507507	0.62507465	0.62506922	0.62500000
0.5	0.66673969	0.66673927	0.66673400	0.66666666
0.6	0.71435481	0.71435439	0.71434937	0.71428571
0.7	0.76929266	0.76929240	0.76928798	0.76923077
0.8	0.83338353	0.83338336	0.83337978	0.83333333
0.9	0.90912169	0.90912184	0.90911972	0.90909090
1.0	1.00000000	1.00000000	1.00000000	1.00000000

CHAPTER 3

A NONASYMPTOTIC METHOD FOR SINGULAR PERTURBATION PROBLEMS

3.1 INTRODUCTION

Instead of replacing the original second order differential equation by an approximate neutral type first order differential equation with a small deviating argument we can replace it simply by an approximate first order differential equation with a small deviating argument. Then also, instead of using trapezoidal formula we can discuss how the higher order methods such as the Simpson's formula can be employed on the resulting problems. However, we would like to preserve the tridiagonal nature of the discrete system.

Based on the above arguments, a nonasymptotic method for solving linear singularly perturbed two point boundary value problems with a boundary layer on the left end of the interval is presented in this chapter. This is an alternative method to that of Chapter 2, and also simpler than that method. This method also does not depend on asymptotic expansions and on the matching of coefficients. It requires a minimum of problem preparation and is readily implemented on a computer. The main feature of this method is that it does not require very fine mesh size. We replace the original second order differential equation by an approximate first order differential equation with

a small deviating argument. Then, we use the Simpson's rule for the numerical integration of the first order differential equation with a small deviating argument to obtain the three-term recurrence relationship. Tridiagonal algebraic system is solved efficiently by employing the discrete invariant imbedding algorithm. The proposed method is to be repeated for different choices of the deviating argument, until the solution profile stabilizes. Several numerical examples have been solved to demonstrate the efficiency of the method.

3.2 NONASYMPTOTIC METHOD

For convenience we call our method the 'nonasymptotic method'. To set the stage for the nonasymptotic method, we consider the following singular perturbation problem (SPP) :

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \; ; \; 0 \leq x \leq 1 \qquad (3.1)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta \qquad (3.2)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

Let δ be a small positive deviating argument $(0 < \delta << 1)$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x - \delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{3.3}$$

and consequently, the equation (3.1) is replaced by the following first order differential equation with a small deviating argument :

$$2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\delta y'(x) + \delta^2 a(x) y'(x) + \delta^2 b(x) y(x) = \delta^2 f(x) \tag{3.4}$$

Transition from the equation (3.1) to equation (3.4) is admitted, because of the condition that δ is small ($0 < \delta << 1$). We rewrite the equation (3.4) in the following convenient form :

$$y'(x) = p(x) y(x-\delta) + q(x) y(x) + r(x) \tag{3.5}$$

for $\delta \leq x \leq 1$ where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)} \tag{3.6}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)} \tag{3.7}$$

$$r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)} . \tag{3.8}$$

We now divide the interval $[0,1]$ into N equal parts with mesh size h,

i.e. $h = \frac{1}{N}$ and $x_i = ih$ for $i = 0, 1, 2, \ldots, N$.

Integrating the equation (3.5) in $[x_i, x_{i+1}]$, ($i=1, 2, \ldots, N-1$), we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} [p(x) y(x-\delta) + q(x) y(x) + r(x)]\, dx.$$

By making use of the Simpson's rule for evaluating the integrals approximately, we obtain

$$
\begin{aligned}
y(x_{i+1})-y(x_i) = {} & \tfrac{h}{6}[p(x_i)y(x_i-\delta)] \\
& + \tfrac{h}{6}\,[4p(x_{i+1/2})y(x_{i+1/2}-\delta)] \\
& + \tfrac{h}{6}\,[p(x_{i+1})y(x_{i+1}-\delta)] \\
& + \tfrac{h}{6}[q(x_i)y(x_i)+4q(x_{i+1/2})y(x_{i+1/2})+q(x_{i+1})y(x_{i+1})] \\
& + \tfrac{h}{6}\,[r(x_i)+4r(x_{i+1/2})+r(x_{i+1})] \,. \qquad (3.9)
\end{aligned}
$$

By means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x)$$

and, then by approximating $y'(x)$ by linear interpolation, we get

$$
\begin{aligned}
y(x_i-\delta) & \approx y(x_i) - \delta\left(\frac{y(x_i) - y(x_{i-1})}{h}\right) \\
& = \left(1 - \frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}\, y(x_{i-1}) \qquad (3.10)
\end{aligned}
$$

and

$$y(x_{i+1}-\delta) \approx \left(1 - \frac{\delta}{h}\right)y(x_{i+1}) + \frac{\delta}{h}\, y(x_i) \qquad (3.11)$$

and, in a similar way

$$
\begin{aligned}
y(x_{i+1/2}-\delta) & \approx y(x_{i+1/2}) - \delta\left(\frac{y(x_{i+1}) - y(x_i)}{h}\right) \\
& = y(x_{i+1/2}) - \frac{\delta}{h}\, y(x_{i+1}) + \frac{\delta}{h}\, y(x_i) \qquad (3.12)
\end{aligned}
$$

Hence, by making use of (3.10), (3.11), and (3.12) in (3.9) leads to

$$y(x_{i+1})-y(x_i) = [\frac{(h-\delta)}{6} p(x_{i+1}) - \frac{4\delta}{6} p(x_{i+1/2}) + \frac{h}{6} q(x_{i+1})]y(x_{i+1})$$

$$+ [\frac{(h-\delta)}{6} p(x_i) + \frac{4\delta}{6} p(x_{i+1/2}) + \frac{\delta}{6} p(x_{i+1}) + \frac{h}{6} q(x_i)]y(x_i)$$

$$+ [\frac{\delta}{6} p(x_i)]y(x_{i-1})$$

$$+ [\frac{4h}{6} p(x_{i+1/2}) + \frac{4h}{6} q(x_{i+1/2})]y(x_{i+1/2})$$

$$+ \frac{h}{6} [r(x_i) + 4r(x_{i+1/2}) + r(x_{i+1})] \qquad (3.13)$$

Again, by using Taylor's theorem it is easy to show that

$$y(x_{i+1/2}) = \frac{y(x_i)+y(x_{i+1})}{2} + \frac{h}{8} [y'(x_i)-y'(x_{i+1})] + O(h^4) \qquad (3.14)$$

In view of the equation (3.5) and the above (3.14) we get

$$y(x_{i+1/2}) \approx \frac{1}{2} y(x_i) + \frac{1}{2} y(x_{i+1})$$

$$+ \frac{h}{8} [p(x_i)y(x_i-\delta)+q(x_i)y(x_i)+r(x_i)]$$

$$- \frac{h}{8}[p(x_{i+1})y(x_{i+1}-\delta)+q(x_{i+1})y(x_{i+1})+r(x_{i+1})]. \qquad (3.15)$$

By making use of (3.15) in (3.13) we get

$$
\begin{aligned}
y(x_{i+1})-y(x_i) = {} & [\tfrac{(h-\delta)}{6}\, p(x_{i+1}) - \tfrac{4\delta}{6}\, p(x_{i+1/2}) + \tfrac{h}{6}\, q(x_{i+1}) \\
& +(\tfrac{1}{2}-\tfrac{h}{8}\, q(x_{i+1}))(\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))]\, y(x_{i+1}) \\
& + [\tfrac{(h-\delta)}{6}\, p(x_i) + \tfrac{4\delta}{6}\, p(x_{i+1/2}) + \tfrac{\delta}{6}\, p(x_{i+1}) + \tfrac{h}{6}\, q(x_i) \\
& +(\tfrac{1}{2}+\tfrac{h}{8}\, q(x_i))(\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))]\; y(x_i) \\
& + [\tfrac{\delta}{6}\, p(x_i)]\; y(x_{i-1}) \\
& + [\tfrac{h}{8}(\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))p(x_i)]\; y(x_i-\delta) \\
& - [\tfrac{h}{8}(\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))p(x_{i+1})]\; y(x_{i+1}-\delta) \\
& + [\tfrac{1}{6} + \tfrac{1}{8}\, (\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))]\; hr(x_i) \\
& + [\tfrac{1}{6} - \tfrac{1}{8}(\tfrac{4h}{6}\, p(x_{i+1/2}) + \tfrac{4h}{6}\, q(x_{i+1/2}))]\; hr(x_{i+1}) \\
& + \tfrac{4h}{6}\, r(x_{i+1/2}). \qquad (3.16)
\end{aligned}
$$

Finally, by making use of equations (3.10) and (3.11) in the equation (3.16) leads after simple manipulation to the following three-term recurrence relationship :

$$E_i\ y_{i-1} - F_i\ y_i + G_i\ y_{i+1} = H_i \qquad (3.17)$$

for $i = 1,2,\ldots,N-1$; where

$$E_i = [-\frac{\delta}{6} p_i - \frac{\delta}{8} (\frac{4h}{6} p_{i+1/2} + \frac{4h}{6} q_{i+1/2}) p_i] \tag{3.18}$$

$$F_i = [1 + \frac{(h-\delta)}{6} p_i + \frac{4\delta}{6} p_{i+1/2} + \frac{\delta}{6} p_{i+1} + \frac{h}{6} q_i$$
$$+ (\frac{1}{2} + \frac{(h-\delta)}{8} p_i - \frac{\delta}{8} p_{i+1} + \frac{h}{8} q_i)(\frac{4h}{6} p_{i+1/2} + \frac{4h}{6} q_{i+1/2})] \tag{3.19}$$

$$G_i = [1 + \frac{4\delta}{6} p_{i+1/2} - \frac{(h-\delta)}{6} p_{i+1} - \frac{h}{6} q_{i+1}$$
$$+ (-\frac{1}{2} + \frac{(h-\delta)}{8} p_{i+1} + \frac{h}{8} q_{i+1})(\frac{4h}{6} p_{i+1/2} + \frac{4h}{6} q_{i+1/2})] \tag{3.20}$$

$$H_i = [\frac{1}{6} + \frac{1}{8} (\frac{4h}{6} p_{i+1/2} + \frac{4h}{6} q_{i+1/2})] hr_i$$
$$+ [\frac{1}{6} - \frac{1}{8} (\frac{4h}{6} p_{i+1/2} + \frac{4h}{6} q_{i+1/2})] hr_{i+1}$$
$$+ \frac{4h}{6} r_{i+1/2} \tag{3.21}$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$, and $r_i = r(x_i)$. Equation (3.17) gives a system of (N-1) equations with (N+1) unknowns y_o to y_N. The two given boundary conditions (3.2) together with these (N-1) equations are then sufficient to solve for the unknowns y_i's. The matrix problem associated with the equation (3.17) is tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient algorithm called Discrete Invariant Imbedding (Angel and Bellman [4]). In this algorithm we set a difference relation of the form

$$y_i = W_i \, y_{i+1} + T_i \tag{3.22}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined. From (3.22) we have

$$y_{i-1} = W_{i-1} \, y_i + T_{i-1}. \tag{3.23}$$

Substituting (3.23) in (3.17), we get

$$y_i = \frac{G_i}{F_i - E_i \, W_{i-1}} \, y_{i+1} + \frac{E_i \, T_{i-1} - H_i}{F_i - E_i \, W_{i-1}} . \tag{3.24}$$

By comparing (3.24) with (3.22), we get

$$W_i = \frac{G_i}{F_i - E_i \, W_{i-1}} \tag{3.25}$$

and

$$T_i = \frac{E_i \, T_{i-1} - H_i}{F_i - E_i \, W_{i-1}} . \tag{3.26}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$; we need to know the initial conditions for W_o and T_o. This can be done by considering the boundary condition $y(0) = \alpha$, as follows

$$y_o = \alpha = W_o \, y_1 + T_o.$$

If we choose $W_o = 0$, then $T_o = \alpha$. Using these initial values, we first compute W_i and T_i for $i = 1, 2, \ldots, N-1$; from (3.25) and (3.26) in the forward process. Then we obtain the solutions y_i for $i = N-1, N-2, \ldots, 2, 1$; in the backward process from (3.22) using the remaining boundary condition $y_N = y(1) = \beta$.

Repeat the process for different choices of δ (deviating argument, satisfying the condition, $0 < \delta << 1$), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|y(x)^{m+1} - y(x)^{m}| \leq \sigma \; ; \; 0 \leq x \leq 1 \qquad (3.27)$$

where

$y(x)^{m}$ = the solution for the m-th iterate of δ

and

σ = prescribed tolerance bound.

3.3 NUMERICAL EXAMPLES

To demonstrate the efficiency of the nonasymptotic method, we have applied it to three examples : a homogeneous SPP, a non-homogeneous SPP, and a SPP with variable coefficients. Each of these examples has been chosen because either analytic or approximate solutions are available for comparison.

Example 3.1 : Firstly, we consider the homogeneous SPP from Bender and Orszag [22], Page : 480, Problem : 9.17 with $\alpha = 0$,

$$\varepsilon y''(x) + y'(x) - y(x) = 0 \; , \; 0 \leq x \leq 1 \qquad (3.28)$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1. \qquad (3.29)$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}}{(e^{m_2} - e^{m_1})} \qquad (3.30)$$

where $m_1 = \frac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \frac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$.

The computational results are presented in the Table 3.1, 3.2, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 3.2 : Secondly, we consider the non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [125], Example : 2;

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad 0 \leq x \leq 1 \tag{3.31}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{3.32}$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon-1)\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}. \tag{3.33}$$

The computational results are presented in the Table 3.3, 3.4, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 3.3 : Finally, we consider the SPP with variable coefficients from Kevorkian and Cole [84], Page : 33; Equations : 2.3.26 and 2.3.27 with $\alpha = -\frac{1}{2}$;

$$\varepsilon y''(x) + (1 - \tfrac{x}{2})y'(x) - \tfrac{1}{2}y(x) = 0, \quad 0 \leq x \leq 1 \tag{3.34}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{3.35}$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [106], Page : 148; Equation : 4.2.32) as our 'exact' solution

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp(-(x - \tfrac{x^2}{4})/\varepsilon). \tag{3.36}$$

The computational results are presented in the Table 3.5, 3.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

3.4 DISCUSSION

First of all, it is of interest to note that a feature of the nonasymptotic method which is different from the method of Chapter 2 is that to evaluate the coefficients (3.18) - (3.21), values of p(x), q(x), and r(x) are required at the grid location $(x_i + \frac{h}{2})$. This is due to the use of Simpson's formula. Another feature of the nonasymptotic method is that to evaluate the coefficients (3.18) - (3.21), we donot need the derivatives of the functions p(x), q(x), and r(x) which is essential in the method of Chapter 2. Since the Simpson's formula is more accurate than the trapezoidal formula, it is expected that the present method will in general produce more accurate results than that of Chapter 2. As mentioned, the method is iterative on the deviating argument δ. The scheme is to be repeated for various choices of δ (deviating argument), until the solution profiles stabilize. The choice of δ is not unique but can assume any number of values satisfying the condition, $0 < \delta << 1$. To reduce the amount of computation, we fix the mesh size h and vary the deviating argument δ. Finally, we pick up the smallest value of δ which produces the required accuracy. We have implemented this method on three examples, a homogeneous SPP, a non-homogeneous SPP and a SPP with variable coefficients, by taking different values for ε. We have tabulated the computational results obtained by the present method as well as the exact solution. We have given here only a few values although the

solutions are computed at all the points with mesh size h. It can be observed from the tables that the present method approximates the exact solution very well. This shows the efficiency and accuracy of the present method.

3.5 CONCLUSIONS

We have described the nonasymptotic method for solving singular perturbation problems. It provides an alternative and supplementary method to the conventional ways of solving singular perturbation problems. The nonasymptotic method possesses several advantages. First, it does not depend on asymptotic expansions and on the matching of the coefficients, and hence it does not require the analysis, experimentation, and knowledge necessary in the conventional methods to find the appropriate asymptotic expansion. Second, it does not require very fine mesh size. Third, the method is primarily a numerical technique and is readily adapted for computer implementation with a modest amount of problem preparation. Fourth, it is accurate and efficient. The numerical experiments demonstrate this fact. Finally, we conclude that the nonasymptotic method appears to be one of the best choices for numerically solving singular perturbation problems with less amount of computational effort.

Table 3.1

Computational results for Example 3.1, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	0.001 y(x)	0.003 y(x)	Exact Solution
0.00	1.00000000	1.00000000	1.00000000
0.02	0.41556128	0.38199903	0.37567774
0.04	0.38651592	0.38336102	0.38325930
0.06	0.39198304	0.39103368	0.39099386
0.08	0.39973213	0.39892365	0.39888451
0.1	0.40777165	0.40697346	0.40693440
0.2	0.45050903	0.44972562	0.44968726
0.3	0.49772628	0.49696886	0.49693177
0.4	0.54989229	0.54917496	0.54913982
0.5	0.60752575	0.60686526	0.60683289
0.6	0.67119969	0.67061586	0.67058726
0.7	0.74154721	0.74106340	0.74103969
0.8	0.81926778	0.81891139	0.81889392
0.9	0.90513413	0.90493722	0.90492758
1.0	1.00000000	1.00000000	1.00000000

Table 3.2

Computational results for Example 3.1, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	0.001 y(x)	0.003 y(x)	Exact Solution
0.00	1.00000000	1.00000000	1.00000000
0.02	0.37608900	0.37542742	0.37533228
0.04	0.38301522	0.38293293	0.38291405
0.06	0.39074936	0.39066783	0.39064895
0.08	0.39864038	0.39855897	0.39854015
0.1	0.40669076	0.40660951	0.40659073
0.2	0.44944793	0.44936812	0.44934966
0.3	0.49670035	0.49662317	0.49660532
0.4	0.54892063	0.54884751	0.54883059
0.5	0.60663104	0.60656369	0.60654812
0.6	0.67040880	0.67034926	0.67033548
0.7	0.74089178	0.74084243	0.74083101
0.8	0.81878495	0.81874859	0.81874018
0.9	0.90486737	0.90484727	0.90484262
1.0	1.00000000	1.00000000	1.00000000

Table 3.3

Computational results for Example 3.2, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ / x	0.001 y(x)	0.003 y(x)	Exact Solution
0.00	0.00000000	0.00000000	0.00000000
0.02	-0.91098335	-0.96744455	-0.97764000
0.04	-0.94811980	-0.95616699	-0.95648000
0.06	-0.92989083	-0.93431020	-0.93452001
0.08	-0.90745152	-0.91155563	-0.91176000
0.1	-0.88399906	-0.88800007	-0.88820001
0.2	-0.75466667	-0.75822229	-0.75840001
0.3	-0.60533334	-0.60844451	-0.60860001
0.4	-0.43600001	-0.43866675	-0.43880001
0.5	-0.24666669	-0.24888897	-0.24900001
0.6	-0.03733337	-0.03911112	-0.03920000
0.7	0.19199996	0.19066659	0.19060000
0.8	0.44133330	0.44044438	0.44039998
0.9	0.71066664	0.71022218	0.71019998
1.0	1.00000000	1.00000000	1.00000000

Table 3.4

Computational results for Example 3.2, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	0.001 y(x)	0.003 y(x)	Exact Solution
0.00	0.00000000	0.00000000	0.00000000
0.02	-0.97790708	-0.97926145	-0.97940400
0.04	-0.95775919	-0.95818662	-0.95820799
0.06	-0.93577359	-0.93619109	-0.93621200
0.08	-0.91298689	-0.91339555	-0.91341600
0.1	-0.88940019	-0.88980000	-0.88982000
0.2	-0.75946671	-0.75982230	-0.75984000
0.3	-0.60953330	-0.60984460	-0.60986000
0.4	-0.43959989	-0.43986688	-0.43988000
0.5	-0.24966651	-0.24988914	-0.24990000
0.6	-0.03973315	-0.03991139	-0.03991999
0.7	0.19020017	0.19006641	0.19006000
0.8	0.44013348	0.44004424	0.44003999
0.9	0.71006677	0.71002211	0.71002001
1.0	1.00000000	1.00000000	1.00000000

Table 3.5

Computational results for Example 3.3, $\varepsilon = 10^{-3}$ and $h = 0.01$

δ → x	0.01 y(x)	0.05 y(x)	Exact Solution
0.00	0.00000000	0.00000000	0.00000000
0.02	0.50476965	0.50507780	0.50505050
0.04	0.51045162	0.51025380	0.51020408
0.06	0.51571275	0.51551378	0.51546392
0.08	0.52108284	0.52088333	0.52083334
0.1	0.52656592	0.52636591	0.52631579
0.2	0.55580799	0.55560612	0.55555556
0.3	0.58848804	0.58828594	0.58823530
0.4	0.62525016	0.62505014	0.62500000
0.5	0.66690987	0.66671540	0.66666666
0.6	0.71451534	0.71433173	0.71428571
0.7	0.76943678	0.76927203	0.76923077
0.8	0.83350027	0.83336679	0.83333333
0.9	0.90919426	0.90911160	0.90909090
1.0	1.00000000	1.00000000	1.00000000

Table 3.6

Computational results for Example 3.3, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	0.01 y(x)	0.05 y(x)	Exact Solution
0.00	0.00000000	0.00000000	0.00000000
0.02	0.50506970	0.50505522	0.50505050
0.04	0.51022895	0.51020904	0.51020408
0.06	0.51548886	0.51546890	0.51546392
0.08	0.52085834	0.52083833	0.52083334
0.1	0.52634086	0.52632080	0.52631579
0.2	0.55558087	0.55556063	0.55555556
0.3	0.58826064	0.58824039	0.58823530
0.4	0.62502508	0.62500502	0.62500000
0.5	0.66669103	0.66667159	0.66666666
0.6	0.71430873	0.71429036	0.71428571
0.7	0.76925144	0.76923491	0.76923077
0.8	0.83335008	0.83333671	0.83333333
0.9	0.90910127	0.90909307	0.90909090
1.0	1.00000000	1.00000000	1.00000000

CHAPTER 4

THE METHOD OF INNER BOUNDARY CONDITION: A NEW APPROACH FOR SOLVING SINGULAR PERTURBATION PROBLEMS

4.1 INTRODUCTION

In order to know the behavior of the solution of the singular perturbation problem in the boundary layer region, it is always suggestive to divide the original problem into two problems and to solve them separately. Keeping this in mind, a new approach based on the method of inner boundary condition for solving singular perturbation problems is presented in this chapter. The general idea of the domain decomposition process dates back to Prandtl [124], which was later named as the method of matched asymptotic expansions (See, e.g. Eckhaus [43], Van Dyke [138], Kevorkian and Cole [84], O'Malley [113], Nayfeh [106], Hemker et al.[66], and Axelsson et al. [12].). Recently, Flaherty and O'Malley [56] have used a similar idea very successfully to treat certain stiff systems of ordinary boundary value problems. Very recently, Hsiao and Jordan [74] have used this idea to certain classes of singular perturbation problems. Also, Lorenz [95] has discussed this approach for a class of nonlinear singular perturbation problems. Looking at the literature cited above, an interesting but amusing observation that has been made is that some of the workers have attempted to solve the singular perturbation problem in the

outer region as a reduced problem obtained by putting $\varepsilon = 0$ and thereby ignoring the contribution due to this term, however small it may be, to the solution of the original problem. Our aim here is to solve the singular perturbation problem, as it is, in both inner as well as outer regions without disturbing the nature of the equation. This method is designed on the basis of the asymptotic behavior of the singular perturbation problem. The original problem is partitioned into inner and outer region differential equation systems. To obtain the terminal boundary condition, asymptotic expansion is used in the outer region with appropriate boundary condition. Using an appropriate transformation, a new inner region problem is obtained and solved as a two point boundary value problem. The derivative boundary condition at the terminal point is then derived from the solution of the inner region problem. Using this condition, the outer region problem is efficiently solved by employing the classical finite difference scheme. Finally, the solutions of inner and outer region problems are combined to obtain an approximate solution to the original problem. The process is to be repeated for various choices of terminal point of the inner region, until the solution profiles stabilize in both the regions. Some test examples have been solved to demonstrate the applicability of the method and the numerical results are compared with exact solutions.

4.2 DESCRIPTION OF THE METHOD

To be specific, we consider the following singular perturbation problem (SPP) :

$$\varepsilon y''(x)+f(x)y'(x)+g(x)y(x) = h(x), 0 \leq x \leq 1 \qquad (4.1)$$

$$\text{with} \quad y(0) = \alpha \text{ and } y(1) = \beta \qquad (4.2)$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$); α, β are given constants; $f(x), g(x)$, and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $f(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

As mentioned, we divide the original problem into two problems, an inner region problem and an outer region problem. Then the inner and outer regions are given by $0 \leq x \leq O(\varepsilon)$ and $O(\varepsilon) \leq x \leq 1$ respectively. Let x_p be the terminal point or common point or width or thickness of the inner region. To obtain the terminal boundary condition (i.e. an approximate value of y at the terminal point x_p), we use the asymptotic expansion in the outer region with appropriate boundary condition. As is well known from the singular perturbation theory (cf. Nayfeh [106]) that for the case $f(x) > 0$ in $[0,1]$, the boundary condition at the origin must be dropped and the boundary condition at the other end (i.e. at $x = 1$) has to be taken into account in the outer region. Hence, the outer region problem is given by

$$\varepsilon y''(x)+f(x)y'(x)+g(x)y(x) = h(x), \; x_p \leq x \leq 1 \qquad (4.3)$$

$$\text{with} \quad y(1) = \beta. \qquad (4.4)$$

We shall seek an outer solution as an asymptotic expansion in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n(x)\, \varepsilon^n \qquad (4.5)$$

where $a_n(x)$ are unknown functions to be determined. Substituting the equation (4.5) in the equations (4.3) and (4.4) we get

$$\begin{aligned} &\varepsilon \left[a_0'' + a_1''\varepsilon + a_2''\varepsilon^2+\ldots\right] \\ &+ f(x) \left[a_0'+a_1'\varepsilon + a_2'\varepsilon^2+\ldots\right] \\ &+ g(x) \left[a_0+a_1\varepsilon+a_2\varepsilon^2+\ldots\right] = h(x) \end{aligned} \qquad (4.6)$$

for $x_p \leq x \leq 1$ with

$$a_0(1)+a_1(1)\varepsilon+a_2(1)\varepsilon^2 +\ldots = \beta. \qquad (4.7)$$

Equating the coefficients of various powers of ε in equations (4.6) and (4.7), we get

$$f(x)a_0'+g(x)a_0 = h(x) \text{ with } a_0(1) = \beta \qquad (4.8)$$

$$f(x)a_n'+g(x)a_n = -a_{n-1}'' \text{ with } a_n(1) = 0 \qquad (4.9)$$

$$\text{where } n = 1, 2, 3, \ldots .$$

The solution of (4.8), if we take account of the boundary condition, is

$$a_0(x) = \left[\exp\left(-\int_1^x \frac{g(\xi)}{f(\xi)}\, d\xi\right)\right]\left[\int_1^x \frac{h(s)}{f(s)} \exp\left(\int_1^s \frac{g(\xi)}{f(\xi)}\, d\xi\right)ds+\beta\right]. \qquad (4.10)$$

Recursively, the functions $a_1(x), a_2(x), \ldots$ can be obtained by solving the equation (4.9) for $n = 1, 2, 3, \ldots$. Thus, the expansion for $y(x)$ given in equation (4.5) is obtained. Hence, the terminal boundary condition can be obtained from (4.5) and denote

$$y(x_p) = \sum_{n=0}^{\infty} a_n(x_p)\varepsilon^n = \bar{\alpha}. \qquad (4.11)$$

Since the terminal point x_p is common to both the inner and outer regions, it defines the inner region problem as a two point boundary value problem :

$$\varepsilon y''(x)+f(x)y'(x)+g(x)y(x) = h(x), \; 0 \le x \le x_p \qquad (4.12)$$

$$\text{with } y(0) = \alpha \text{ and } y(x_p) = \bar{\alpha}. \qquad (4.13)$$

We choose the transformation

$$x = t\varepsilon \qquad (4.14a)$$

to create a new inner region problem. By rescaling the equation (4.12) with

$$y(x) = y(t\varepsilon) = Y(t) \qquad (4.14b)$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \qquad (4.14c)$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \qquad (4.14d)$$

$$f(x) = f(t\varepsilon) = F(t) \qquad (4.14e)$$

$$g(x) = g(t\varepsilon) = G(t) \qquad (4.14f)$$

$$h(x) = h(t\varepsilon) = H(t), \qquad (4.14g)$$

we obtain the new differential equation for the inner region solution as,

$$Y''(t)+F(t)Y'(t)+\varepsilon G(t)Y(t) = \varepsilon H(t). \qquad (4.15)$$

Boundary conditions for the equation (4.15) are determined by (4.14b), (4.14a) and (4.13) as,

$$Y(0) = \alpha \text{ and } Y(t_p) = \bar{\alpha}. \qquad (4.16)$$

We solve this new inner region problem (4.15) - (4.16) to obtain the solution over the interval $0 \leq t \leq t_p$. From this solution, we determine the value of $Y'(t_p)$ and in turn $y'(x_p)$ by using the equation (4.14c) and denote it as

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}. \qquad (4.17)$$

Returning back to the outer region, we have the outer region problem as a two point boundary value problem :

$$\varepsilon y''(x)+f(x)y'(x)+g(x)y(x) = h(x), \; x_p \leq x \leq 1 \qquad (4.18)$$

$$\text{with } y'(x_p) = \bar{\beta} \text{ and } y(1) = \beta. \qquad (4.19)$$

We solve this outer region problem (4.18) - (4.19) by employing the classical finite difference scheme to obtain the solutions over the interval $x_p \leq x \leq 1$. We divide the interval $[x_p, 1]$ into N equal subintervals with step size $h = \frac{1-x_p}{N}$, and replace the differential equation (4.18) by a set of difference equations using the central difference formulae (cf. Fox [57])

$$y_i' \cong \frac{y_{i+1}-y_{i-1}}{2h}, \tag{4.20a}$$

$$y_i'' \cong \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}. \tag{4.20b}$$

The derivative boundary condition (4.19a) is also replaced by the corresponding difference equation. Including the difference equation at the terminal point x_p, we have N linear algebraic equations involving $y(x_p), y(x_1), \ldots, y(x_{N-1})$ as unknowns. This algebraic system is in the tridiagonal form, which can be very easily and efficiently solved by a direct method (for details, see Angel and Bellman [4], Conte and De Boor [35]).

After solving both the inner and the outer region problems, we combine the solutions of inner and outer region problems to obtain an approximate solution to the original problem (4.1) - (4.2) over the interval $0 \leq x \leq 1$.

Repeat the process for different choices of 'x_p' (terminal point of the inner region), until the solution profiles do not differ materially from iteration to iteration. For computations, one may use an absolute error criteria, namely

$$|Y(t)^{(m+1)}-Y(t)^{(m)}| \leq \sigma \; ; \; 0 \leq t \leq t_p \tag{4.21}$$

where

$$Y(t)^{(m)} = \text{mth iteration of inner region solution}$$

and

$$\sigma = \text{prescribed tolerance bound.}$$

NOTE : As mentioned, x_p is the terminal point of the inner region, $0 \leq x \leq O(\varepsilon)$. That is $x_p = O(\varepsilon)$. Hence, in our numerical experimentation, we started with $x_p = \varepsilon$ and repeated the process by increasing the x_p (as 5ε, 10ε, 20ε) till the condition (4.21) is satisfied.

4.3 TEST EXAMPLES AND NUMERICAL RESULTS

Example 4.1 : Consider the following homogeneous SPP which has earlier been solved by Reinhardt [125] and Roberts [128].

$$\varepsilon y'' + y' + y = 0; \quad 0 \leq x \leq 1 \tag{4.22a}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 2. \tag{4.22b}$$

The exact solution is given by

$$y_\varepsilon(x) = \frac{(2 - e^{r_2})}{(e^{r_1} - e^{r_2})} e^{r_1 x} + \frac{(e^{r_1} - 2)}{(e^{r_1} - e^{r_2})} e^{r_2 x}$$

where $r_1 = \dfrac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon}$ and $r_2 = \dfrac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon}$.

In the outer region $x_p \leq x \leq 1$, the problem becomes

$$\varepsilon y'' + y' + y = 0 ; \quad x_p \leq x \leq 1$$

$$\text{with } y(1) = 2.$$

Assuming the solution in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n(x) \varepsilon^n$$

we get the first order problem as follows

$$a_o' + a_o = 0 \text{ with } a_o(1) = 2$$

$$a_n' + a_n + a_{n-1}'' = 0 \text{ with } a_n(1) = 0.$$

By taking only three terms in the expansion, we get

$$a_o(x) = 2e^{1-x}$$

$$a_1(x) = 2(1-x)e^{1-x}$$

$$a_2(x) = 2\left(\frac{x^2}{2} - 3x + \frac{5}{2}\right)e^{1-x}$$

and hence $y(x) = 2e^{1-x} + 2\varepsilon(1-x)e^{1-x} + 2\varepsilon^2\left(\frac{x^2}{2} - 3x + \frac{5}{2}\right)e^{1-x}.$

Evaluate $y(x)$ at $x = x_p$ and denote $y(x_p) = \bar{\alpha}$.

By choosing the transformation $x = t\varepsilon$ and by rescaling, we get a new differential equation in the inner region

$$Y'' + Y' + \varepsilon Y = 0 \; ; \; 0 \leq t \leq t_p$$

$$\text{with } Y(0) = \alpha \text{ and } Y(t_p) = \bar{\alpha}.$$

This two point boundary value problem has analytical solution

$$Y(t) = \frac{(\bar{\alpha} - e^{p_2 t_p})}{(e^{p_1 t_p} - e^{p_2 t_p})} e^{p_1 t} + \frac{(e^{p_1 t_p} - \bar{\alpha})}{(e^{p_1 t_p} - e^{p_2 t_p})} e^{p_2 t}$$

where $p_1 = \dfrac{-1 + \sqrt{1-4\varepsilon}}{2}$ and $p_2 = \dfrac{-1 - \sqrt{1-4\varepsilon}}{2}$.

From this $Y(t)$, we can find $Y'(t)$ which will provide us $y'(x_p)$ and we denote

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}.$$

Now coming to outer region again, we have

$$\varepsilon y'' + y' + y = 0 \; ; \; x_p \le x \le 1$$

$$\text{with } y'(x_p) = \bar{\beta} \text{ and } y(1) = \beta.$$

This two point boundary value problem is solved using finite difference scheme and the numerical solutions for different values of ε are presented in Tables 4.1 and 4.2.

Example 4.2 : Consider the following non-homogenous SPP which arises frequently in fluid dynamics. This has earlier been solved by Reinhardt [125]

$$\varepsilon y'' + y' = 1+2x \; ; \; 0 \le x \le 1, \tag{4.23a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{4.23b}$$

The exact solution is given by

$$y_\varepsilon(x) = \frac{(2\varepsilon-1)(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))} + x(x+1-2\varepsilon).$$

In the outer region, the problem becomes

$$\varepsilon y'' + y' = 1+2x \; ; \; x_p \le x \le 1$$

$$\text{with } y(1) = 1.$$

Assuming the solution $y(x)$ in the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x)\, \varepsilon^n$$

we get the first order equations

From this $Y(t)$, we can find $Y'(t)$ which will provide us $y'(x_p)$ and we denote

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}.$$

Now coming to outer region again, we have

$$\varepsilon y'' + y' = 1+2x \;, \quad x_p \leq x \leq 1$$

$$\text{with } y'(x_p) = \bar{\beta} \text{ and } y(1) = 1.$$

This two point boundary value problem is solved using finite difference scheme and the numerical solutions for different values of ε are presented in the Tables 4.3 and 4.4.

4.4 DISCUSSION

In the present method, the solution of the outer region problem provides the terminal condition $(y(x_p))$ for the inner region problem. And in turn, the solution of the inner region problem provides the terminal condition $(y'(x_p))$ for the outer region problem. This serves as the link between the two regions. As mentioned, the method is iterative on the terminal point of the inner region. The process is to be repeated for various choices of x_p (terminal point of the inner region) until the solution profiles stabilize in both the regions. The point x_p is not unique but can assume a number of values. To reduce the amount of computation, we choose the smallest value of x_p which produces the required accuracy. Because the inner region problem interval is very small relative to the entire interval

of the original problem, we can usually improve our accuracy by making x_p larger. As an alternative to the solution of the outer region problem (4.18) - (4.19a-b) we may use the solution (4.5) of the problem (4.3) - (4.4) over the interval $x_p \leq x \leq 1$. We have implemented the present method on two test examples by taking different values for ε. Since x_p is the terminal point of the inner region, that is $x_p = O(\varepsilon)$, in our numerical experimentation, we started with $x_p = \varepsilon$ and repeated the process by increasing the x_p (as $5\varepsilon, 10\varepsilon, 20\varepsilon$) till the condition (4.21) is satisfied. We have tabulated the numerical results obtained by the present method as well as the exact solution. The numerical experimentation on these examples demonstrates that the present method approximates the exact solution well.

4.5 CONCLUSIONS

We have described a new approach based on the method of inner boundary condition for solving singular perturbation problems. It is a practical method and can easily be implemented on a computer to solve such problems. We have illustrated the method with two examples with known solutions and have demonstrated that the present method approximates the exact solution well.

Table 4.1

Numerical results for Example 4.1, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	Exact solution
0.0	1.0000000	1.0000000	1.0000000	1.0000000
$2.5(10^{-4})$	2.5528358	1.9803880	1.9803425	1.9803425
$5.0(10^{-4})$	3.7620445	2.7438053	2.7437245	2.7437244
$1.0(10^{-3})$	5.4365691	3.8009368	3.8008070	3.8008070
$5.0(10^{-3})$		5.3849677	5.3847644	5.3847643
$1.0(10^{-2})$		5.3878108	5.3876073	5.3876072
$2.0(10^{-2})$			5.3341478	5.3341478
$3.0(10^{-2})$				5.2810193
$4.0(10^{-2})$				5.2284199
$1.0(10^{-1})$	4.9236615	4.9235989	4.9235755	4.9236444
$2.0(10^{-1})$	4.4546650	4.4545913	4.4545907	4.4546513
$3.0(10^{-1})$	4.0303420	4.0302835	4.0302835	4.0303313
$4.0(10^{-1})$	3.6464375	3.6463921	3.6463921	3.6464292
$5.0(10^{-1})$	3.2991012	3.2990671	3.2990671	3.2990950
$6.0(10^{-1})$	2.9848500	2.9848253	2.9848253	2.9848455
$7.0(10^{-1})$	2.7005325	2.7005156	2.7005156	2.7005293
$8.0(10^{-1})$	2.4432971	2.4432869	2.4432869	2.4432951
$9.0(10^{-1})$	2.2105642	2.2105596	2.2105596	2.2105633
1.0	2.0000000	2.0000000	2.0000000	2.0000000

Table 4.2

Numerical results for Example 4.1, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	Exact solution
0.0	1.0000000	1.0000000	1.0000000	1.0000000
$2.5(10^{-5})$	2.5525396	1.9813069	1.9812623	1.9812624
$5.0(10^{-5})$	3.7616268	2.7455385	2.7454593	2.7454593
$1.0(10^{-4})$	5.4365637	3.8042071	3.8040798	3.8040798
$1.0(10^{-3})$		5.4316725	5.4314709	5.4314709
$2.0(10^{-3})$			5.4262430	5.4262430
$3.0(10^{-3})$				5.4208189
$4.0(10^{-3})$				5.4154003
$1.0(10^{-1})$	4.9190618	4.9196396	4.9196401	4.9196491
$2.0(10^{-1})$	4.4509658	4.4514307	4.4514308	4.4514380
$3.0(10^{-1})$	4.0274189	4.0277814	4.0277818	4.0277874
$4.0(10^{-1})$	3.6441669	3.6444516	3.6444520	3.6444563
$5.0(10^{-1})$	3.2973860	3.2976039	3.2976043	3.2976075
$6.0(10^{-1})$	2.9836106	2.9837662	2.9837665	2.9837688
$7.0(10^{-1})$	2.6996907	2.6997969	2.6997971	2.6997986
$8.0(10^{-1})$	2.4427893	2.4428533	2.4428535	2.4428544
$9.0(10^{-1})$	2.2103645	2.2103634	2.2103636	2.2103639
1.0	2.0000000	2.0000000	2.0000000	2.0000000

Table 4.3

Numerical results for Example 4.2, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	1 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$2.5(10^{-4})$	-0.34898259	-0.22051727	-0.22050725	-0.22050726
$5.0(10^{-4})$	-0.62071514	-0.39220097	-0.39218314	-0.39218315
$1.0(10^{-3})$	-0.99700100	-0.62988596	-0.62985732	-0.62985732
$5.0(10^{-3})$		-0.98630553	-0.98626053	-0.98626053
$1.0(10^{-2})$		-0.98792000	-0.98787469	-0.98787469
$2.0(10^{-2})$			-0.97764000	-0.97764000
$3.0(10^{-2})$				-0.96716000
$4.0(10^{-2})$				-0.95643000
$1.0(10^{-1})$	-0.88820002	-0.88820860	-0.88819764	-0.88820001
$2.0(10^{-1})$	-0.75840002	-0.75840015	-0.75839998	-0.75840001
$3.0(10^{-1})$	-0.60860002	-0.60860004	-0.60860004	-0.60860001
$4.0(10^{-1})$	-0.43880005	-0.43880006	-0.43880007	-0.43880001
$5.0(10^{-1})$	-0.24900011	-0.24900003	-0.24900008	-0.24900001
$6.0(10^{-1})$	-0.03920017	-0.03920008	-0.03920008	-0.03920000
$7.0(10^{-1})$	0.19059982	0.19059993	0.19059993	0.19060000
$8.0(10^{-1})$	0.44039984	0.44039994	0.44039994	0.44039998
$9.0(10^{-1})$	0.71019990	0.71019997	0.71019997	0.71019999
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 4.4

Numerical results for Example 4.2, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$2.5(10^{-5})$	-0.34983702	-0.22114003	-0.22112999	-0.22112999
$5.0(10^{-5})$	-0.62228481	-0.39335851	-0.39334065	-0.39334064
$1.0(10^{-4})$	-0.99970001	-0.63192284	-0.63189416	-0.63189415
$1.0(10^{-3})$		-0.99879920	-0.99875381	-0.99875381
$2.0(10^{-3})$			-0.99779640	-0.99779639
$3.0(10^{-3})$				-0.99679159
$4.0(10^{-3})$				-0.99578480
$1.0(10^{-1})$	-0.88982894	-0.88982006	-0.88982006	-0.88982000
$2.0(10^{-1})$	-0.75984900	-0.75984006	-0.75984006	-0.75984000
$3.0(10^{-1})$	-0.60986901	-0.60986005	-0.60986005	-0.60986000
$4.0(10^{-1})$	-0.43988874	-0.43988006	-0.43988006	-0.43988000
$5.0(10^{-1})$	-0.24990813	-0.24990012	-0.24990012	-0.24990000
$6.0(10^{-1})$	-0.03992718	-0.03992017	-0.03992017	-0.03991999
$7.0(10^{-1})$	0.19005410	0.19005981	0.19005981	0.19006000
$8.0(10^{-1})$	0.44003573	0.44003983	0.44003983	0.44003999
$9.0(10^{-1})$	0.71001770	0.71001990	0.71001990	0.71002001
1.0	1.00000000	1.00000000	1.00000000	1.00000000

CHAPTER 5

A BOUNDARY LAYER TECHNIQUE FOR A CLASS OF SINGULAR PERTURBATION PROBLEMS

5.1 INTRODUCTION

In this chapter, we propose a boundary layer technique for numerically solving a class of linear singularly perturbed two point boundary value problems in ordinary differential equations with a boundary layer on the left end of the underlying interval. This technique is designed on the basis of the asymptotic behaviour of the singular perturbation problems. As in Chapter 4, the original problem is divided into inner and outer region problems. However asymptotic expansions are not employed. To obtain the terminal boundary condition, the reduced problem is solved with an appropriate boundary condition. Then, a new inner region problem is created and solved as a two point boundary value problem. In turn, the outer region problem is also modified by replacing it by an approximate first order differential equation of neutral type with a small deviating argument. Then the new outer region problem is efficiently treated by employing the trapezoidal formula coupled with discrete invariant imbedding. Finally, the solutions of inner and outer region problems are combined to obtain an approximate solution to the original problem. The process is to be repeated for various choices of the terminal point of the inner region until the profiles stabilize in both the regions. Some numerical experiments have been included to demonstrate the efficiency of the present method.

5.2 BOUNDARY LAYER TECHNIQUE

For convenience we call our method the 'boundary layer technique'. In the interest of clarity, we restrict our attention to the following singular perturbation problem (SPP),

$$\varepsilon y''(x)+a(x)y'(x)-b(x)y(x) = f(x), \; 0 \leq x \leq 1 \qquad (5.1)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta, \qquad (5.2)$$

where ε is a small positive parameter ($0 < \varepsilon << 1$); α, β are given constants; $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$ and $b(x) \geq 0$, $a(x) \geq M > 0$ on $[0,1]$, where M is some positive constant. Under these assumptions, (5.1) - (5.2) has a unique solution $y(x)$ which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

As mentioned, we divide the original problem into two problems, an inner region problem and an outer region problem.

5.2.1 TERMINAL BOUNDARY CONDITION :- Let x_p be the terminal point or common point or width or thickness of the inner region. To obtain the terminal boundary condition (i.e. an approximate value of y at the terminal point x_p), we solve the reduced problem with an appropriate boundary condition. It has been shown by Nayfeh [106] that for the case $a(x) > 0$ on $[0,1]$, the boundary condition at the origin must be dropped and the boundary condition at the other end (i.e. at $x = 1$) has to be taken into account in the outer region. Hence by setting $\varepsilon = 0$, we will have the reduced problem :

$$a(x)y'(x) - b(x)y(x) = f(x) \tag{5.3}$$

$$\text{with } y(1) = \beta. \tag{5.4}$$

The solution of (5.3), using the boundary condition (5.4), is

$$y(x) = \left[\exp\left(\int_1^x \frac{b(\zeta)}{a(\zeta)}\, d\zeta\right)\right]\left[\int_1^x \frac{f(s)}{a(s)} \exp\left(\int_s^1 \frac{b(\zeta)}{a(\zeta)}\, d\zeta\right) ds + \beta\right] \tag{5.5}$$

Hence, the terminal boundary condition can be obtained from (5.5), and denote

$$y(x_p) = \bar{\alpha}. \tag{5.6}$$

5.2.2 INNER REGION PROBLEM :- Since the terminal point is common to both the inner and outer regions, it defines the inner region problem as a two point boundary value problem :

$$\varepsilon y''(x)+a(x)y'(x) - b(x)y(x) = f(x), \; 0 \le x \le x_p \tag{5.7a}$$

$$\text{with } y(0) = \alpha \text{ and } y(x_p) = \bar{\alpha}. \tag{5.7b}$$

We choose the transformation

$$x = \varepsilon t \tag{5.8}$$

to create a new differential equation for the inner region solution. Using (5.8), rescale equation (5.7a) with,

$$y(x) = y(t\varepsilon) = Y(t) \tag{5.9a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{5.9b}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{5.9c}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{5.9d}$$

$$b(x) = b(t\varepsilon) = B(t) \tag{5.9e}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{5.9f}$$

to obtain the new differential equation as follows

$$Y''(t) + A(t)Y'(t) - \varepsilon B(t)Y(t) = \varepsilon F(t). \tag{5.10}$$

Boundary conditions for (5.10) are determined by (5.8), (5.9a) and (5.7b)

$$Y(0) = \alpha \text{ and } Y(t_p) = \bar{\alpha}. \tag{5.11}$$

The new differential equation (5.10) is further modified by expanding $A(t)$, $B(t)$ and $F(t)$ in Taylor series and dropping the terms of $O(\varepsilon)$, to obtain

$$Y''(t) + A(0)\, Y'(t) = 0. \tag{5.12}$$

Now by making use of the assumption made earlier that $a(x) \geq M > 0$ on $[0,1]$ where M is some positive constant, we get $A(0) = m > 0$ where m is some positive constant. Hence it is clear that the equation (5.12) is a constant coefficient differential equation. We remark that for the inner region problem the differential equation always has constant coefficients and it is immaterial whether or not $a(x)$ and $b(x)$ are constants. Thus, we will carry $A(0)$ throughout the discussion instead of replacing it by m, the constant given above. We solve this modified inner region problem given by (5.12) along with boundary conditions given by (5.11) to obtain the solutions over the interval $0 \leq t \leq t_p$.

The most general solution to (5.12) is given by

$$Y(t) = C_1 + C_2 \exp(-A(0)t) \tag{5.13}$$

where C_1 and C_2 are constants to be determined by satisfying the boundary conditions (5.11). Note that $Y(t)$, the solution (5.13) of the inner region problem (5.12) remains bounded as $t \to +\infty$. The constants C_1 and C_2 are determined by imposing the boundary conditions (5.11) on $Y(t)$, as

$$C_2 = \frac{\bar{\alpha}-\alpha}{(\exp(-A(0)t_p)-1)} \tag{5.14a}$$

and

$$C_1 = \alpha - C_2. \tag{5.14b}$$

This completes the estimates for the solution of the inner region problem.

5.2.3 OUTER REGION PROBLEM :- Returning to the outer region, we obtain an approximate differential equation for the outer region solution as follows:

Let us denote $x_p = \delta$ (this is only for our convenience and notational simplicity) and then it is clear that $0 < \delta << 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{5.15}$$

and, consequently, the equation (5.1) is replaced by the following first order differential equation with a small deviating argument, in the outer region,

$$\varepsilon y'(x) - \varepsilon y'(x-\delta) + \delta a(x) y'(x) - \delta b(x) y(x) = \delta f(x) \quad (5.16)$$

for $\delta \leq x \leq 1$ with the boundary conditions

$$y(\delta) = \bar{\alpha} \text{ and } y(1) = \beta. \quad (5.17)$$

Transition from the equation (5.1) to equation (5.16) is admitted, because of the condition that δ is small $(0 < \delta << 1)$. Rearrangement of the equation (5.16) is sometimes called the differential equation of 'neutral type' with a small deviating argument, namely

$$y'(x) = p(x) y'(x-\delta) + q(x)\; y(x) + r(x) \quad (5.18)$$

for $\delta \leq x \leq 1$ where

$$p(x) = \frac{\varepsilon}{\varepsilon + \delta a(x)} \quad (5.19a)$$

$$q(x) = \frac{\delta b(x)}{\varepsilon + \delta a(x)} \quad (5.19b)$$

$$r(x) = \frac{\delta f(x)}{\varepsilon + \delta a(x)}. \quad (5.19c)$$

Now, we describe the method for numerically solving the approximated outer region problem given by the equation (5.18) along with the boundary conditions given by (5.17). As usual, we divide the interval $[\delta, 1]$ into N equal parts with mesh points $\delta = x_0 < x_1 < x_2 < \dots < x_N = 1$ and the mesh size $h = \frac{1-\delta}{N}$, $x_i = \delta + ih$ for $i = 0, 1, 2, \dots, N$. By integrating the equation (5.18) in $[x_i, x_{i+1}]$, $(i = 1, \dots, N-1)$ we get

$$y(x_{i+1})-y(x_i) = \int_{x_i}^{x_{i+1}} [p(x)y'(x-\delta)+q(x)y(x) + r(x)]\, dx$$

$$= p(x_{i+1})y(x_{i+1}-\delta) - p(x_i)y(x_i-\delta)$$

$$+ \int_{x_i}^{x_{i+1}} [-p'(x)y(x-\delta)+q(x)y(x)+r(x)]\, dx.$$

By making use of trapezoidal formula for evaluating the integrals approximately, we obtain

$$\begin{aligned} y(x_{i+1})-y(x_i) &= [p(x_{i+1}) - \tfrac{h}{2}\, p'(x_{i+1})]\; y(x_{i+1}-\delta) \\ &\quad -[p(x_i) + \tfrac{h}{2}\, p'(x_i)]\; y(x_i-\delta) \\ &\quad + \tfrac{h}{2}\, [q(x_{i+1})y(x_{i+1})+q(x_i)y(x_i)] \\ &\quad + \tfrac{h}{2}\, [r(x_{i+1}) + r(x_i)]. \end{aligned} \tag{5.20}$$

Again, by means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) \tag{5.21}$$

and, then by using the linear interpolation, we get

$$\begin{aligned} y(x_i-\delta) &\approx y(x_i) - \delta\left(\frac{y(x_i) - y(x_{i-1})}{h}\right) \\ &= (1 - \tfrac{\delta}{h})\; y(x_i) + \tfrac{\delta}{h}\; y(x_{i-1}) \end{aligned} \tag{5.22a}$$

and

$$y(x_{i+1}-\delta) \approx (1 - \tfrac{\delta}{h})\; y(x_{i+1}) + \tfrac{\delta}{h}\; y(x_i). \tag{5.22b}$$

Hence, by making use of (5.22a-b) in (5.20) leads, after simple calculation to the final three-term recurrence relationship, namely

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{5.23}$$

for $i = 1, 2, \ldots, N-1$; where

$$E_i = \frac{\delta}{h}\left[p_i + \frac{h}{2} p_i'\right] \tag{5.24a}$$

$$F_i = 1 + \frac{\delta}{h}\left[p_{i+1} - \frac{h}{2} p_{i+1}'\right] - \left(1 - \frac{\delta}{h}\right)\left[p_i + \frac{h}{2} p_i'\right] + \frac{h}{2} q_i \tag{5.24b}$$

$$G_i = 1 - \left(1 - \frac{\delta}{h}\right)\left[p_{i+1} - \frac{h}{2} p_{i+1}'\right] - \frac{h}{2} q_{i+1} \tag{5.24c}$$

$$H_i = \frac{h}{2}\left[r_{i+1} + r_i\right], \tag{5.24d}$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Equation (5.23) gives a system of (N-1) equations with (N+1) unknowns y_0 to y_N. The two given boundary conditions (5.17) together with these (N-1) equations are then sufficient to solve for the unknowns y_i's. The solution of this tridiagonal system can easily be obtained by employing an efficient algorithm called 'Discrete Invariant Imbedding'. In this algorithm we set a difference relation of the form

$$y_i = W_i\, y_{i+1} + T_i \tag{5.25}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined. From (5.25) we have

$$y_{i-1} = W_{i-1}\, y_i + T_{i-1}. \tag{5.26a}$$

Substituting (5.26a) in (5.23), we get

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}}\, y_{i+1} + \frac{E_i\, T_{i-1} - H_i}{F_i - E_i\, W_{i-1}}. \tag{5.26b}$$

By comparing (5.26b) with (5.25), we get

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{5.27}$$

and

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} . \tag{5.28}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$, we need to know the initial conditions for W_o and T_o. This can be done by considering the boundary condition $y(\delta) = \bar{\alpha}$, as follows

$$y_o = \bar{\alpha} = W_o\, y_1 + T_o .$$

Choose $W_o = 0$ and $T_o = \bar{\alpha}$. Using these initial values, we first compute W_i and T_i for $i = 1, 2, \ldots, N-1$; from (5.27) and (5.28) in the forward process. Then we obtain the solutions y_i for $i = N-1, N-2, \ldots, 2, 1$; in the backward process from (5.25) using the remaining boundary condition $y_N = \beta$. And this completes the estimates for the solutions of the outer region problem.

5.2.4 SOLUTION OF THE ORIGINAL PROBLEM :- After solving both the inner and outer region problems by employing the methods described in the previous sections, we combine the solutions of inner and outer region problems to obtain an approximate solution to the original problem (5.1) - (5.2) over the interval $0 \leq x \leq 1$.

Repeat the process for different choices of x_p (terminal point of the inner region), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|Y(t)^{m+1} - Y(t)^m| \leq \sigma \; ; \; 0 \leq t \leq t_p \tag{5.29}$$

where

$$Y(t)^m = m^{th} \text{ iteration of inner region solution}$$

and

$$\sigma = \text{prescribed tolerance bound.}$$

5.3 NUMERICAL EXPERIMENTS

In this section, we present, three numerical experiments to demonstrate the efficiency of the boundary layer technique.

Example 5.1 : Firstly, we consider the homogeneous SPP from Bender and Orszag [22], Page : 480, Problem : 9.17 with $\alpha = 0$;

$$\varepsilon y''(x) + y'(x) - y(x) = 0, 0 \leq x \leq 1 \tag{5.30a}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1. \tag{5.30b}$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2} - 1)\, e^{m_1 x} + (1-e^{m_1})e^{m_2 x}}{(e^{m_2} - e^{m_1})}$$

where $m_1 = \dfrac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \dfrac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$.

The computational results are presented in Table 5.1, 5.2, for $\varepsilon = 10^{-3}$, 10^{-4}, respectively.

Example 5.2 : Secondly, we consider the non-homogeneous SPP from Fluid dynamics for fluid of small viscosity, Reinhardt [125], Example : 2;

$$\varepsilon y''(x) + y'(x) = 1 + 2x, \quad 0 \le x \le 1 \tag{5.31a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{5.31b}$$

The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}.$$

The computational results are presented in the Table 5.3, 5.4, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 5.3 : Finally, we consider the SPP with variable coefficients from Kevorkian and Cole [84], Page : 33; Equations : 2.3.26 and 2.3.27 with $\alpha = -\frac{1}{2}$;

$$\varepsilon y''(x) + (1 - \tfrac{x}{2})y'(x) - \tfrac{1}{2}y(x) = 0, \quad 0 \le x \le 1 \tag{5.32a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{5.32b}$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [106], Page : 148 ; Equation : 4.2.32) as our 'exact' solution

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp(-(x - \frac{x^2}{4})/\varepsilon).$$

The computational results are presented in the Table 5.5, 5.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

5.4 DISCUSSION

As mentioned, the method is iterative on the terminal point of the inner region. The process is to be repeated for different choices of x_p (terminal point of the inner region), until the solution profiles stabilize in both the regions. As an alternative to the solution of the outer region problem (5.18) - (5.17), we may use the solution of the reduced problem (5.3) - (5.4) over the interval $x_p \leq x \leq 1$. But for more accurate results we prefer to solve the outer region problem (5.18) - (5.17) as it is. We have implemented this method on three problems, a homogeneous SPP, a non-homogeneous SPP and a SPP with variable coefficients, by taking different values for ε. In the tables, the underlined value indicates that it is a terminal boundary condition obtained by solving the reduced problem and the corresponding x is the terminal point x_p. It can be observed from the tables that the present method approximates the exact solutions very well.

5.5 CONCLUSIONS

We have described the boundary layer technique for solving a class of linear singular perturbation problems. It is a practical method, easily implemented on a computer to solve singular perturbation problems with a modest amount of problem preparation. We have illustrated the method with three examples with known solutions and have demonstrated that the boundary layer technique approximates the exact solution well.

Table 5.1

Computational results for Example 5.1, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	1 y(x)	5 y(x)	10 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$0.25(10^{-3})$	0.77892958	0.85963757	0.86098694	0.86022566
$0.50(10^{-3})$	0.60675977	0.75032320	0.75272348	0.75141688
$0.75(10^{-3})$	0.47267377	0.66518907	0.66840779	0.66671805
$0.10(10^{-2})$	0.36824750	0.59888655	0.60274267	0.60079141
$0.25(10^{-2})$		0.41753507	0.42313463	0.42089517
$0.50(10^{-2})$		0.36972344	0.37578264	0.37432568
$0.75(10^{-2})$			0.37189575	0.37136243
$0.10(10^{-1})$			0.37157669	0.37197179
$0.20(10^{-1})$				0.37567774
0.1	0.40692899	0.40692633	0.40692936	0.40693440
0.2	0.44968197	0.44967934	0.44968231	0.44968726
0.3	0.49692666	0.49692412	0.49692697	0.49693177
0.4	0.54913503	0.54913259	0.54913528	0.54913982
0.5	0.60682848	0.60682619	0.60682873	0.60683289
0.6	0.67058333	0.67058133	0.67058358	0.67058726
0.7	0.74103644	0.74103475	0.74103664	0.74103969
0.8	0.81889152	0.81889030	0.81889167	0.81889392
0.9	0.90492624	0.90492558	0.90492634	0.90492758
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 5.2

Computational results for Example 5.1, $\varepsilon = 10^{-4}$

t_p → x	1 $y(x)$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$0.25(10^{-4})$	0.77881365	0.85926788	0.86025050	0.86017700
$0.50(10^{-4})$	0.60655357	0.74966560	0.75141349	0.75128750
$0.75(10^{-4})$	0.47239726	0.66430724	0.66665112	0.66648842
$0.10(10^{-3})$	0.36791623	0.59783009	0.60062813	0.60045052
$0.25(10^{-3})$		0.41600098	0.42007858	0.41986551
$0.50(10^{-3})$		0.36806343	0.37247573	0.37234153
$0.75(10^{-3})$			0.36856825	0.36852598
$0.10(10^{-2})$			0.36824750	0.36829735
$0.20(10^{-2})$				0.36863712
0.1	0.40670876	0.40657252	0.40660961	0.40659073
0.2	0.44946563	0.44933177	0.44936819	0.44934966
0.3	0.49671744	0.49658803	0.49662325	0.49660532
0.4	0.54893680	0.54881427	0.54884759	0.54883059
0.5	0.60664593	0.60653307	0.60656374	0.60654812
0.6	0.67042193	0.67032222	0.67034931	0.67033548
0.7	0.74090261	0.74082013	0.74084245	0.74083101
0.8	0.81879306	0.81873204	0.81874862	0.81874018
0.9	0.90487186	0.90483817	0.90484730	0.90484262
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 5.3

Computational results for Example 5.2, $\varepsilon = 10^{-3}$

t_p → x	1 y(x)	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.25(10^{-3})	-0.34958173	-0.22158070	-0.21897505	-0.22050726
0.50(10^{-3})	-0.62183624	-0.39414790	-0.38951298	-0.39218315
0.75(10^{-3})	-0.83386829	-0.52854340	-0.52232807	-0.52582913
0.10(10^{-2})	-0.99899900	-0.63321070	-0.62576456	-0.62985732
0.25(10^{-2})		-0.91949801	-0.90868531	-0.91357792
0.50(10^{-2})		-0.99497500	-0.98327474	-0.98626053
0.75(10^{-2})			-0.98939742	-0.98990677
0.10(10^{-1})			-0.98990000	-0.98787469
0.20(10^{-1})				-0.97764000
0.1	-0.88820004	-0.88820004	-0.88820000	-0.88820001
0.2	-0.75840004	-0.75840004	-0.75840005	-0.75840001
0.3	-0.60860004	-0.60860004	-0.60860009	-0.60860001
0.4	-0.43880007	-0.43880007	-0.43880019	-0.43880001
0.5	-0.24900012	-0.24900012	-0.24900023	-0.24900001
0.6	-0.03920018	-0.03920018	-0.03920024	-0.03920000
0.7	0.19059981	0.19059981	0.19059977	0.19060000
0.8	0.44039983	0.44039983	0.44039981	0.44039998
0.9	0.71019989	0.71019989	0.71019987	0.71019998
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 5.4

Computational results for Example 5.2, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 y(x)	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$0.25(10^{-4})$	−0.34989701	−0.22258836	−0.22098783	−0.22112999
$0.50(10^{-4})$	−0.62239707	−0.39594033	−0.39309332	−0.39334064
$0.75(10^{-4})$	−0.83462034	−0.53094700	−0.52712922	−0.52745293
$0.10(10^{-3})$	<u>−0.99989999</u>	−0.63609029	−0.63151648	−0.63189415
$0.25(10^{-3})$		−0.92367953	−0.91703780	−0.91748140
$0.50(10^{-3})$		<u>−0.99949975</u>	−0.99231289	−0.99256325
$0.75(10^{-3})$			−0.99849179	−0.99849661
$0.10(10^{-2})$			<u>−0.99899900</u>	−0.99875381
$0.20(10^{-2})$				−0.99779639
0.1	−0.88982894	−0.88981283	−0.88982007	−0.88982000
0.2	−0.75984900	−0.75985117	−0.75984006	−0.75984000
0.3	−0.60986901	−0.60988609	−0.60986006	−0.60986000
0.4	−0.43988874	−0.43991945	−0.43988007	−0.43988000
0.5	−0.24990813	−0.24994677	−0.24990013	−0.24990000
0.6	−0.03992718	−0.03996922	−0.03992018	−0.03991999
0.7	0.19005410	0.19001377	0.19005980	0.19006000
0.8	0.44003573	0.44000257	0.44003982	0.44003999
0.9	0.71001770	0.70999858	0.71001989	0.71002001
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 5.5

Computational results for Example 5.3, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ / x	1 y(x)	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$0.25(10^{-3})$	0.17505353	0.11162895	0.11116043	0.11065603
$0.50(10^{-3})$	0.31138535	0.19856566	0.18773226	0.19684075
$0.75(10^{-3})$	0.41756069	0.26627205	0.26515448	0.26397108
$0.10(10^{-2})$	0.50025012	0.31900183	0.31766295	0.31626441
$0.25(10^{-2})$		0.46322898	0.46128476	0.45951910
$0.50(10^{-2})$		0.50125313	0.49914932	0.49786303
$0.75(10^{-2})$			0.50225744	0.50160159
$0.10(10^{-1})$			0.50251257	0.50248929
$0.20(10^{-1})$				0.50505050
0.1	0.52706302	0.52706544	0.52705756	0.52631579
0.2	0.55631009	0.55631222	0.55630407	0.55555556
0.3	0.58899113	0.58899312	0.58898475	0.58823530
0.4	0.62574834	0.62575012	0.62574168	0.62500000
0.5	0.66739447	0.66739603	0.66738757	0.66666666
0.6	0.71497302	0.71497417	0.71496628	0.71428571
0.7	0.76984834	0.76984816	0.76984129	0.76923077
0.8	0.83383425	0.83383410	0.83382790	0.83333333
0.9	0.90940138	0.90940142	0.90939704	0.90909090
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 5.6

Computational results for Example 5.3, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 $y(x)$	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$0.25(10^{-4})$	0.17497475	0.11137772	0.11065996	0.11060525
$0.50(10^{-4})$	0.31124522	0.19811878	0.19684202	0.19674528
$0.75(10^{-4})$	0.41737277	0.26567278	0.26396068	0.26383216
$0.10(10^{-3})$	<u>0.50002500</u>	0.31828390	0.31623274	0.31608069
$0.25(10^{-3})$		0.46218645	0.45920794	0.45901360
$0.50(10^{-3})$		<u>0.50012503</u>	0.49690202	0.49675395
$0.75(10^{-3})$			0.49999614	0.49991064
$0.10(10^{-2})$			<u>0.50025012</u>	0.50022737
$0.20(10^{-2})$				0.50050050
0.1	0.52633590	0.52633239	0.52639127	0.52631579
0.2	0.55557548	0.55557548	0.55563169	0.55555556
0.3	0.58826146	0.58825888	0.58831152	0.58823530
0.4	0.62502872	0.62502611	0.62507539	0.62500000
0.5	0.66669772	0.66669563	0.66674002	0.66666666
0.6	0.71431752	0.71431600	0.71435499	0.71428571
0.7	0.76926211	0.76926092	0.76929309	0.76923077
0.8	0.83336072	0.83335999	0.83338386	0.83333333
0.9	0.90910897	0.90910910	0.90912221	0.90909090
1.0	1.00000000	1.00000000	1.00000000	1.00000000

CHAPTER 6

NUMERICAL SOLUTION OF SINGULAR PERTURBATION PROBLEMS BY TERMINAL BOUNDARY VALUE TECHNIQUES

6.1 INTRODUCTION

It is always interesting to look for a technique or modification for the singular perturbation problem so that any suitable existing standard numerical method can be employed on the modified problem. Keeping this in mind, we have proposed and illustrated in this chapter two terminal boundary value techniques for numerically solving linear singular perturbation problems. As usual, the original problem is divided into outer and inner region problems. Two techniques are introduced to obtain the terminal boundary condition in the implicit form. Then, the outer region problem is solved as a two point boundary value problem (TPBVP) and an explicit terminal boundary condition is obtained. In turn, a new inner region problem is obtained and solved as a TPBVP, using the explicit terminal boundary condition. Finally, we combine the solutions of both the outer and inner region problems to obtain an approximate solution to the original problem. The process is to be repeated for various choices of the terminal point of the inner region, until the solution profiles stabilize in both the regions. Some numerical examples have been solved to demonstrate the applicability of these techniques.

6.2 DESCRIPTION OF THE METHOD

To be specific, we consider the following singular perturbation problem (SPP) :

$$\varepsilon y''(x)+a(x)y'(x) + b(x)y(x) = f(x), \; 0 \leq x \leq 1 \qquad (6.1)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta \qquad (6.2)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$ and $b(x) \leq 0$, $a(x) \geq M > 0$ on $[0,1]$ where M is some positive constant. Under these assumptions, (6.1) - (6.2) has a unique solution $y(x)$ which, in general, displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

As mentioned, we divide the original singular perturbation problem into two problems, an outer region problem and an inner region problem.

6.2.1 TERMINAL BOUNDARY CONDITION :- Let x_p be the terminal point or common point or width or thickness of the inner region. We propose two techniques for obtaining the terminal boundary condition in the implicit form as follows :

<u>Technique I</u> : Let us denote $x_p = \delta$ (This is only for our convenience and notational simplicity.), and then it is clear that $0 < \delta << 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \qquad (6.3)$$

and, consequently, the equation (6.1) is replaced by the following first order differential equation with a small deviating argument :

$$2\varepsilon y(x-\delta)-2\varepsilon y(x)+2\varepsilon\delta y'(x)+\delta^2 a(x)y'(x)+\delta^2 b(x)y(x) = \delta^2 f(x) \quad (6.4)$$

for $\delta \le x \le 1$. Transition from the equation (6.1) to equation (6.4) is admitted, because of the condition that δ is small ($0 < \delta << 1$). We rewrite the equation (6.4) in the convenient form as follows,

$$[\delta^2 a(x)+2\varepsilon\delta]\, y'(x)+[\delta^2 b(x)-2\varepsilon]\, y(x) = [\delta^2 f(x)-2\varepsilon y(x-\delta)] \quad (6.5)$$

for $\delta \le x \le 1$. We simply evaluate the equation (6.5) at $x = \delta$, and denote

$$p = \delta^2 a(\delta) + 2\varepsilon\delta \quad (6.6)$$

$$q = \delta^2 b(\delta) - 2\varepsilon \quad (6.7)$$

$$\begin{aligned} r &= \delta^2 f(\delta) - 2\varepsilon y(0) \\ &= \delta^2 f(\delta) - 2\varepsilon\alpha \quad \text{since } y(0) = \alpha \end{aligned} \quad (6.8)$$

then the equation (6.5) can be written in the form :

$$py'(\delta) + qy(\delta) = r. \quad (6.9)$$

We take this equation (6.9) as the 'terminal boundary condition' and note that it is in the implicit form.

Remark 6.1 : It can be easily verified that the equation (6.9) is nothing but the equation (6.1) evaluated at $x = \delta$ and $y''(\delta)$ is replaced by using equation (6.3) at $x = \delta$.

Technique II : It is well known from the standard singular perturbation theory (see e.g., O'Malley [113], Nayfeh [106]) that the reduced equation (that is equation (6.1) with $\varepsilon = 0$) is valid in the outer region, and for the case $a(x) > 0$ on $[0,1]$, the outer region is $[x_p, 1]$. Hence by setting $\varepsilon = 0$ in (6.1), we will have the 'reduced equation' :

$$a(x)y'(x)+b(x)y(x) = f(x) \tag{6.10}$$

for $x_p \leq x \leq 1$. We simply evaluate the equation (6.10) at $x = x_p$ and denote

$$p = a(x_p) \tag{6.11}$$

$$q = b(x_p) \tag{6.12}$$

$$r = f(x_p) \tag{6.13}$$

and then rewrite the equation (6.10) in the form :

$$py'(x_p) + qy(x_p) = r. \tag{6.14}$$

We take this equation (6.14) as the 'terminal boundary condition' and note that it is in the implicit form.

Remark 6.2 : It can be easily observed that the Technique II is more simpler and easier than the Technique I.

6.2.2 OUTER REGION PROBLEM :- Using the terminal boundary condition (6.9) or (6.14) which is in the implicit form, we get the outer region problem as a two point boundary value problem :

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{6.15}$$

for $x_p \leq x \leq 1$ with

$$py'(x_p) + qy(x_p) = r, \tag{6.16a}$$

and $$y(1) = \beta. \tag{6.16b}$$

We solve this outer region problem (6.15) - (6.16a-b) to obtain the solutions over the interval $x_p \leq x \leq 1$.

Hence, the solution of the outer region problem will provide the explicit terminal boundary condition. Let us denote it by

$$y(x_p) = \bar{\alpha}. \tag{6.17}$$

6.2.3 INNER REGION PROBLEM :- Since the terminal point is common to both the inner and outer regions, we can formulate the inner region problem as a two point boundary value problem, as follows

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x) = f(x), \quad 0 \leq x \leq x_p \tag{6.18}$$

$$\text{with } y(0) = \alpha \text{ and } y(x_p) = \bar{\alpha} \tag{6.19}$$

where $\bar{\alpha} = y(x_p)$ is obtained from the solution of the outer region problem.

We choose the transformation

$$x = t\varepsilon \tag{6.20}$$

to create a new differential equation for the inner region solution. Using (6.20), rescale (6.18) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{6.21a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{6.21b}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{6.21c}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{6.21d}$$

$$b(x) = b(t\varepsilon) = B(t) \tag{6.21e}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{6.21f}$$

to obtain the new differential equation as follows

$$Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t). \tag{6.22}$$

Boundary conditions for the equation (6.22) are determined by (6.20), (6.21a) and (6.19)

$$Y(0) = \alpha \text{ and } Y(t_p) = \bar{\alpha} . \tag{6.23}$$

We solve this new inner region problem (6.22) - (6.23) to obtain the solutions over the interval $0 \leq t \leq t_p$.

In order to solve the two point boundary value problems given by the equations (6.15) - (6.16a-b) [outer region problem] and (6.22) - (6.23) [inner region problem], we make use of the second order central finite difference scheme (see for details, Fox [57].). In fact, any standard analytical or numerical method can be used.

6.2.4 SOLUTION OF THE ORIGINAL PROBLEM :- After solving both the outer and inner region problems, we combine the solutions of the outer and inner region problems to obtain an approximate solution to the original problem (6.1) - (6.2) over the interval $0 \leq x \leq 1$.

Repeat the process for various choices of x_p (terminal point of the inner region), until the solution profiles donot differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|Y^{(m+1)}(t) - Y^{(m)}(t)| \leq \sigma \; ; \; 0 \leq t \leq t_p \tag{6.24}$$

where

$$Y^{(m)}(t) = m^{th} \text{ iteration of the inner region solution}$$

and

$$\sigma = \text{prescribed tolerance bound.}$$

6.3 NUMERICAL EXAMPLES

In this section, we will discuss three numerical examples to demonstrate the applicability of the method described in the previous sections. Firstly, we use the Technique I for obtaining terminal boundary condition and apply the method on three examples.

Example 6.1 : Consider the homogeneous SPP from Bender and Orszag [22], Page : 480; Problem : 9.17 with $\alpha = 0$;

$$\varepsilon y''(x)+y'(x)-y(x) = 0, \; 0 \leq x \leq 1 \tag{6.25a}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1. \tag{6.25b}$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2}-1)e^{m_1 x} + (1-e^{m_1})\, e^{m_2 x}}{(e^{m_2} - e^{m_1})} \tag{6.26}$$

where $m_1 = \dfrac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \dfrac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$.

The computational results for Example 6.1 with Technique I are presented in the Table 6.1, 6.2, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 6.2 : Now consider the non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [125], Example: 2,

$$\varepsilon y''(x) + y'(x) = 1+2x \, , \; 0 \le x \le 1 \qquad (6.27a)$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \qquad (6.27b)$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon-1)\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}. \qquad (6.28)$$

The computational results for Example 6.2 with Technique I are presented in the Table 6.3, 6.4, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 6.3 : Finally, consider the SPP with variable coefficients from Kevorkian and Cole [84], Page : 33; Equations : 2.3.26 and 2.3.27 with $\alpha = -\frac{1}{2}$;

$$\varepsilon y''(x)+(1-\tfrac{x}{2})y'(x) - \tfrac{1}{2}\, y(x) = 0, \; 0 \le x \le 1 \qquad (6.29a)$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \qquad (6.29b)$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [106], Page : 148; Equation : 4.2.32) as our 'exact' solution

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp(-(x-\tfrac{x^2}{4})/\varepsilon). \qquad (6.30)$$

The computational results for Example 6.3 with Technique I are presented in the Table 6.5, 6.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Now, we use the Technique II for obtaining the terminal boundary condition and then apply the method on the same three examples.

Example 6.1 : $\varepsilon y''(x) + y'(x) - y(x) = 0, \; 0 \leq x \leq 1$ (6.31a)

with $y(0) = 1$ and $y(1) = 1$. (6.31b)

Using Technique II, the terminal boundary condition in the implicit form is given by

$$y'(x_p) - y(x_p) = 0. \tag{6.32}$$

First, we solve the outer region problem as a TPBVP :

$$\varepsilon y''(x) + y'(x) - y(x) = 0 \tag{6.33a}$$

for $x_p \leq x \leq 1$ with

$$y'(x_p) - y(x_p) = 0, \tag{6.33b}$$

$$y(1) = 1. \tag{6.33c}$$

In turn, by using the scaling

$$t = x/\varepsilon \tag{6.33d}$$

we get the inner region problem as a TPBVP :

$$Y''(t) + Y'(t) - \varepsilon Y(t) = 0, \; 0 \leq t \leq t_p \tag{6.34a}$$

$$\text{with } Y(0) = 1 \text{ and } Y(t_p) = \bar{\alpha} \tag{6.34b}$$

where the $\bar{\alpha} = y(x_p)$ is obtained from the outer region solution. The computational results for Example 6.1 with Technique II are presented in the Table 6.7, 6.8, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 6.2 : $\varepsilon y''(x) + y'(x) = 1+2x, \; 0 \le x \le 1$ (6.35a)

with $y(0) = 0$ and $y(1) = 1.$ (6.35b)

Using Technique II, the terminal boundary condition is given by

$$y'(x_p) = 1+2x_p \tag{6.36}$$

First, we solve the outer region problem as a TPBVP :

$$\varepsilon y''(x) + y'(x) = 1+2x \tag{6.37a}$$

for $x_p \le x \le 1$ with

$$y'(x_p) = 1+2x_p \, , \tag{6.37b}$$

$$y(1) = 1. \tag{6.37c}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as TPBVP :

$$Y''(t) + Y'(t) = \varepsilon(1+2\varepsilon t), \; 0 \le t \le t_p \tag{6.38a}$$

$$\text{with } Y(0) = 0 \text{ and } Y(t_p) = \bar{\alpha} \tag{6.38b}$$

where the $\bar{\alpha} = y(x_p)$ is obtained from the outer region solution. The computational results for Example 6.2 with Technique II are presented in the Table 6.9, 6.10, for $\varepsilon = 10^{-3}$, 10^{-4}, respectively.

Example 6.3 : $\varepsilon y''(x)+(1 - \frac{x}{2})y'(x) - \frac{1}{2}\, y(x) = 0, \; 0 \le x \le 1$ (6.39a)

with $y(0) = 0$ and $y(1) = 1.$ (6.39b)

Using Technique II, the terminal boundary condition in the implicit form is given by

$$(1 - \frac{x_p}{2})y'(x_p) - \frac{1}{2}\, y(x_p) = 0. \tag{6.40}$$

First we solve the outer region problem as a TPBVP :

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}\, y(x) = 0 \tag{6.41a}$$

for $x_p \leq x \leq 1$ with

$$(1 - \frac{x_p}{2})y'(x_p) - \frac{1}{2}\, y(x_p) = 0\ , \tag{6.41b}$$

$$y(1) = 1. \tag{6.41c}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as a TPBVP :

$$Y''(t) + (1 - \frac{t\varepsilon}{2})Y'(t) - \frac{\varepsilon}{2}\, Y(t) = 0,\ 0 \leq t \leq t_p \tag{6.42a}$$

$$\text{with } Y(0) = 0 \text{ and } Y(t_p) = \bar{\alpha} \tag{6.42b}$$

where $\bar{\alpha} = y(x_p)$ is obtained from the outer region solution. The computational results for Example 6.3 with Technique II are presented in the Table 6.11, 6.12 for $\varepsilon = 10^{-3}$, 10^{-4} respectively.

6.4 DISCUSSION

The present method is similar in sprit to those suggested in Chapters 4 and 5 but differs in detail. In fact, the methods differ in how they use the data which are available. As mentioned, the present method is iterative on the terminal point of the inner region. The process is to be repeated for various choices of x_p (terminal point of the inner region),

until the solution profiles stabilize in both the regions. To solve the outer and inner region problems, we have used the second order central finite difference scheme (cf. Fox [57]). In fact any standard analytical or numerical method can be used. We have implemented this method on three examples, by taking different values for ε. In the tables, the underlined value indicates that it is an explicit terminal boundary condition obtained by solving the outer region problem, and the corresponding x is the terminal point x_p. It can be observed from the tables that the Technique II is more accurate and superior than the Technique I. We have already seen that the Technique II is more simpler and easier than the Technique I. Hence, the Technique II is preferable for obtaining the terminal boundary condition.

6.5 CONCLUSIONS

We have described two terminal boundary value techniques for solving linear singular perturbation problems. These provide a new approach for solving singular perturbation problems with any suitable existing standard numerical method. Both the techniques have been tested on three examples. Numerical results are compared with the corresponding exact solutions. It is observed that the accuracy predicted can always be achieved with very little computational effort.

Table 6.1

Computational results for Example 6.1 with Technique I, $\varepsilon = 10^{-3}$

t_p → x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$5.0(10^{-4})$	0.75983142	0.75428453	0.75207994	0.75141688
$1.0(10^{-3})$	0.61437242	0.60546149	0.60191987	0.60079141
$2.5(10^{-3})$	0.44080066	0.42785462	0.42270926	0.42089517
$5.0(10^{-3})$	<u>0.39600842</u>	0.38197214	0.37639346	0.37432568
$7.5(10^{-3})$		0.37909239	0.37346537	0.37136243
$1.0(10^{-2})$		<u>0.37972854</u>	0.37408463	0.37197179
$2.0(10^{-2})$			<u>0.37782015</u>	0.37567774
0.1	0.40693456	0.40693456	0.40693456	0.40693440
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.2

Computational results for Example 6.1 with Technique I, $\varepsilon = 10^{-4}$

t_p → x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$5.0(10^{-5})$	0.75972088	0.75416569	0.75194849	0.75128750
$1.0(10^{-4})$	0.61406025	0.60513695	0.60157546	0.60045052
$2.5(10^{-4})$	0.43979859	0.42684373	0.42167313	0.41986551
$5.0(10^{-4})$	<u>0.39401305</u>	0.37999399	0.37439865	0.37234153
$7.5(10^{-4})$		0.37624515	0.37061371	0.36852598
$1.0(10^{-3})$		<u>0.37602583</u>	0.37039011	0.36829735
$2.0(10^{-3})$			<u>0.37073873</u>	0.36863712
0.1	0.40654523	0.40654523	0.40654523	0.40659073
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.5

Computational results for Example 6.3 with Technique I, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.19041222	0.19481503	0.19657557	0.19684075
$1.0(10^{-3})$	0.30588757	0.31296045	0.31578867	0.31626441
$2.5(10^{-3})$	0.44430950	0.45458303	0.45869111	0.45951910
$5.0(10^{-3})$	<u>0.48130878</u>	0.49387429	0.49688800	0.49786303
$7.5(10^{-3})$		0.49612409	0.50060757	0.50160159
$1.0(10^{-2})$		<u>0.49700116</u>	0.50149256	0.50248929
$2.0(10^{-2})$			<u>0.50405163</u>	0.50505050
0.1	0.52707017	0.52707017	0.52707017	0.52631579
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.6

Computational results for Example 6.3 with Technique I, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.19006734	0.19446039	0.19621187	0.19674528
$1.0(10^{-4})$	0.30530420	0.31236074	0.31517413	0.31608069
$2.5(10^{-4})$	0.44323175	0.45347623	0.45756063	0.45901360
$5.0(10^{-4})$	<u>0.47959975</u>	0.49068482	0.49510435	0.49675395
$7.5(10^{-4})$		0.49379200	0.49823951	0.49991064
$1.0(10^{-3})$		<u>0.49410458</u>	0.49855490	0.50022737
$2.0(10^{-3})$			<u>0.49883246</u>	0.50050050
0.1	0.52635260	0.52635260	0.52635260	0.52631579
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.7

Computational results for Example 6.1 with Technique II, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ / x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$5.0(10^{-4})$	0.74958894	0.75124349	0.75125138	0.75141688
$1.0(10^{-3})$	0.59791813	0.60057613	0.60058881	0.60079141
$2.5(10^{-3})$	0.41689544	0.42075704	0.42077545	0.42089517
$5.0(10^{-3})$	0.37009004	0.37427685	0.37429681	0.37432568
$7.5(10^{-3})$		0.37133043	0.37135056	0.37136243
$1.0(10^{-2})$		0.37194327	0.37196346	0.37197179
$2.0(10^{-2})$			0.37567762	0.37567774
0.1	0.40693456	0.40693456	0.40693456	0.40693440
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.8

Computational results for Example 6.1 with Technique II, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ / x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000	1.00000000
$5.0(10^{-5})$	0.74942838	0.75109590	0.75110459	0.75128750
$1.0(10^{-4})$	0.59752741	0.60020595	0.60021990	0.60045052
$2.5(10^{-4})$	0.41579618	0.41968488	0.41970514	0.41986551
$5.0(10^{-4})$	0.36803890	0.37224705	0.37226897	0.37234153
$7.5(10^{-4})$		0.36844823	0.36847028	0.36852598
$1.0(10^{-3})$		0.36822299	0.36824505	0.36829735
$2.0(10^{-3})$			0.36859142	0.36863712
0.1	0.40654523	0.40654523	0.40654423	0.40659073
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.9

Computational results for Example 6.2 with Technique II, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ / x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	-0.39509023	-0.39245303	-0.39243376	-0.39218315
$1.0(10^{-3})$	-0.63442752	-0.63019145	-0.63016049	-0.62985732
$2.5(10^{-3})$	-0.91993885	-0.91378940	-0.91374448	-0.91357792
$5.0(10^{-3})$	**-0.99298652**	-0.98633335	-0.98628476	-0.98626053
$7.5(10^{-3})$		-0.98995624	-0.98990734	-0.98990677
$1.0(10^{-2})$		**-0.98792154**	-0.98787261	-0.98787469
$2.0(10^{-2})$			**-0.97764152**	-0.97764000
0.1	-0.88820003	-0.88820003	-0.88820003	-0.88820001
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.10

Computational results for Example 6.2 with Technique II, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ / x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	-0.39625589	-0.39361392	-0.39359465	-0.39334064
$1.0(10^{-4})$	-0.63647729	-0.63223355	-0.63220260	-0.63189415
$2.5(10^{-4})$	-0.92386067	-0.91770007	-0.91765513	-0.91748140
$5.0(10^{-4})$	**-0.99930880**	-0.99264358	-0.99259496	-0.99256325
$7.5(10^{-4})$		-0.99855359	-0.99850466	-0.99849661
$1.0(10^{-3})$		**-0.99880815**	-0.99875920	-0.99875381
$2.0(10^{-3})$			**-0.99780536**	-0.99779639
0.1	-0.88982894	-0.88982894	-0.88982894	-0.88982000
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.11

Computational results for Example 6.3 with Technique II, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.19859678	0.19726724	0.19725594	0.19684075
$1.0(10^{-3})$	0.31903564	0.31689981	0.31688166	0.31626441
$2.5(10^{-3})$	0.46340742	0.46030506	0.46027369	0.45951910
$5.0(10^{-3})$	<u>0.50199706</u>	0.49863634	0.49860778	0.49786303
$7.5(10^{-3})$		0.50236898	0.50234023	0.50160159
$1.0(10^{-2})$		<u>0.50325708</u>	0.50322830	0.50248929
$2.0(10^{-2})$			<u>0.50579621</u>	0.50505050
0.1	0.52707017	0.52707017	0.52707017	0.52631579
1.0	1.00000000	1.00000000	1.00000000	1.00000000

Table 6.12

Computational results for Example 6.3 with Technique II, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	20 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.19821549	0.19689285	0.19688181	0.19674528
$1.0(10^{-4})$	0.31839253	0.31626798	0.31625025	0.31608069
$2.5(10^{-4})$	0.46223300	0.45914866	0.45912291	0.45901360
$5.0(10^{-4})$	<u>0.50016010</u>	0.49682268	0.49679481	0.49675395
$7.5(10^{-4})$		0.49996873	0.49994067	0.49991064
$1.0(10^{-3})$		<u>0.50028521</u>	0.50025711	0.50022737
$2.0(10^{-3})$			<u>0.50053559</u>	0.50050050
0.1	0.52635260	0.52635260	0.52635260	0.52631579
1.0	1.00000000	1.00000000	1.00000000	1.00000000

CHAPTER 7

AN APPROXIMATE METHOD FOR SOLVING A CLASS OF SINGULAR PERTURBATION PROBLEMS

7.1 INTRODUCTION

In this chapter, we propose and illustrate an approximate method for the numerical solution of a class of linear singularly perturbed two point boundary value problems. It consists of the following steps : (1) The given region is divided into inner and outer regions. (2) The original second-order problem is replaced by an asymptotically equivalent first order problem and solved as an initial value problem in the inner region. (3) A terminal boundary condition is then obtained from the solution of the inner region problem. (4) In turn, an outer region problem is obtained, by replacing the second order differential equation by an approximate first order differential equation with a small deviating argument, and solved efficiently by employing the trapezoidal formula coupled with discrete invariant imbedding algorithm. (5) Finally, the solutions of inner and outer region problems are combined to obtain an approximate solution to the original problem. (6) The process is to be repeated for various choices of the terminal point of the inner region problem, until the solution profiles stabilize in both the regions. Several numerical examples have been solved to demonstrate the applicability of

the method. Finally, the method is extended to a more general class of problems. Again one numerical example is solved in this general class.

7.2 DESCRIPTION OF THE METHOD

To fix the ideas, we consider a class of singular perturbation problems of the form :

$$\varepsilon y''(x) + [a(x)y(x)]' = h(x); \; 0 \leq x \leq 1 \qquad (7.1)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta, \qquad (7.2)$$

where ε is a small positive parameter ($0 < \varepsilon << 1$); α, β are given constants; $a(x)$ and $h(x)$ are assumed to be sufficiently continuously differentiable function in $[0,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

As mentioned the method consists of the following steps :

Step 1 : Divide the original region into two regions, an inner region and an outer region. Let x_p be the terminal point or width or thickness of the inner region. Then the inner and outer regions are given by $0 \leq x \leq x_p$ and $x_p \leq x \leq 1$ respectively.

Step 2 : Replace the original second order problem by an asymptotically equivalent first order problem as follows :

By integrating the equation (7.1), we obtain

$$\varepsilon y'(x) + a(x)y(x) = f(x) + K \qquad (7.3)$$

where $$f(x) = \int h(x)\, dx$$

and K is a constant to be determined.

In order to determine the constant K, we introduce the condition that the reduced equation of (7.3) should satisfy the boundary condition, $y(1) = \beta$.

$$\text{i.e. } y(1) = \frac{1}{a(1)}\left[f(1) + K\right] = \beta$$

$$\therefore \quad K = a(1)\beta - f(1). \tag{7.4}$$

Remark 7.1 : This choice of K ensures that the solution of the reduced problem of (7.1) satisfies the reduced equation of (7.3).

Now, we adjoin the condition (which we drop, whenever we formulate the reduced problem of the equation (7.1)) $y(0) = \alpha$ to the equation (7.3) to obtain a first order problem as follows :

$$\varepsilon y'(x) + a(x)y(x) = f(x) + K \tag{7.5}$$

$$y(0) = \alpha, \tag{7.6}$$

where the constant K is given by the equation (7.4).

Thus in a manner of speaking, we have replaced the original second order problem (7.1) - (7.2) with the asymptotically equivalent first order problem (7.5) - (7.6).

We choose the transformation

$$t = x/\varepsilon \tag{7.7}$$

to create a new differential equation. Using (7.7), rescale the equation (7.5) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{7.8a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{7.8b}$$

$$a(x) = a(t\varepsilon) = A(t) \tag{7.8c}$$

$$f(x) = f(t\varepsilon) = F(t), \tag{7.8d}$$

to obtain the new differential equation as follows

$$Y'(t) + A(t)Y(t) = F(t) + K. \tag{7.9}$$

Initial condition for (7.9) is determined by (7.7), (7.8a) and (7.6)

$$Y(0) = \alpha. \tag{7.10}$$

Theoretically the differential equation (7.9) can be solved over the entire interval $0 \leq t \leq 1/\varepsilon$ with the initial condition (7.10). Practically, the interval $[0, 1/\varepsilon]$ becomes unreasonably large as $\varepsilon \to 0$ so we limit the range to $[0, t_p]$, where $t_p = \frac{x_p}{\varepsilon} << \frac{1}{\varepsilon}$. Hence, it leads the inner region problem as an initial value problem :

$$Y'(t) + A(t)Y(t) = F(t) + K \tag{7.11}$$

$$\text{for } 0 \leq t \leq t_p \text{ with } Y(0) = \alpha. \tag{7.12}$$

We solve this inner region problem (7.11) - (7.12) to obtain the solutions over the interval $0 \leq t \leq t_p$.

The analytical solution of (7.11), using the initial condition (7.12), is given by

$$Y(t) = \left[\exp\left(-\int_0^t A(\xi)d\xi\right)\right]\left[\int_0^t (F(s)+K)\exp\left(\int_0^s A(\xi)d\xi\right)ds + \alpha\right] \tag{7.13}$$

Step 3 : Obtain the terminal boundary condition from the equation (7.13) and denote

$$Y(t_p) = \bar{\alpha}. \tag{7.14}$$

Hence from the equations (7.7), (7.8a) and (7.14), we get

$$y(x_p) = \bar{\alpha}. \tag{7.15}$$

Step 4 : Obtain the outer region problem as follows :

Let us denote $x_p = \delta$ and then it is clear that $0 < \delta << 1$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{7.16}$$

and consequently, the equation (7.1) is replaced by the following first order differential equation with a small deviating argument:

$$2\varepsilon y(x-\delta) - 2\varepsilon y(x) + 2\varepsilon\delta y'(x) + \delta^2 a(x) y'(x) + \delta^2 a'(x) y(x) = \delta^2 h(x) \tag{7.17}$$

for $\delta \leq x \leq 1$ with the boundary conditions

$$y(\delta) = \bar{\alpha} \text{ and } y(1) = \beta. \tag{7.18}$$

We rewrite the equation (7.17) in the following convenient form :

$$y'(x) = p(x) y(x-\delta) + q(x) y(x) + r(x) \tag{7.19}$$

for $\delta \leq x \leq 1$ where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)} \tag{7.20a}$$

$$q(x) = \frac{2\varepsilon - \delta^2 a'(x)}{2\varepsilon\delta + \delta^2 a(x)} \tag{7.20b}$$

$$r(x) = \frac{\delta^2 h(x)}{2\varepsilon\delta + \delta^2 a(x)} . \tag{7.20c}$$

Now, we describe the method for numerically solving the outer region problem given by the equation (7.19) with the boundary conditions given by (7.18). We divide the interval $[\delta, 1]$ into N equal parts with mesh size h,

i.e. $h = \frac{1-\delta}{N}$ and $x_i = \delta + ih$ for $i = 0, 1, 2, \ldots, N$.

By integrating the equation (7.19) in $[x_i, x_{i+1}]$, $(i=1, 2, \ldots, N-1.)$ we get

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} [p(x)y(x-\delta) + q(x)y(x) + r(x)]\, dx.$$

By making use of the trapezoidal formula for evaluating the integrals approximately, we obtain

$$\begin{aligned} y(x_{i+1}) - y(x_i) &= \frac{h}{2}[p(x_{i+1})y(x_{i+1}-\delta) + p(x_i)y(x_i-\delta)] \\ &+ \frac{h}{2}[q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i)] \\ &+ \frac{h}{2}[r(x_{i+1}) + r(x_i)] . \end{aligned} \tag{7.21}$$

Again, by means of Taylor series expansion, we have

$$y(x-\delta) \approx y(x) - \delta y'(x)$$

and, then by approximating $y'(x)$ by linear interpolation, we get

$$\begin{aligned} y(x_i-\delta) &\approx y(x_i) - \delta\left(\frac{y(x_i)-y(x_{i-1})}{h}\right) \\ &= \left(1 - \frac{\delta}{h}\right)y(x_i) + \frac{\delta}{h}\, y(x_{i-1}) \end{aligned} \tag{7.22a}$$

and $$y(x_{i+1}-\delta) \approx (1-\frac{\delta}{h})\; y(x_{i+1}) + \frac{\delta}{h}\, y(x_i). \tag{7.22b}$$

Hence, by making use of (7.22a-b) in (7.21) leads after simple manipulation to the final three-term recurrence relationship, namely

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{7.23}$$

for $i = 1,2,\ldots,N-1$; where

$$E_i = -\frac{\delta}{2}\, p_i \tag{7.24a}$$

$$F_i = 1 + \frac{\delta}{2}\, p_{i+1} + \frac{h}{2}\,(1-\frac{\delta}{h})p_i + \frac{h}{2}\, q_i \tag{7.24b}$$

$$G_i = 1 - \frac{h}{2}\,(1-\frac{\delta}{h})p_{i+1} - \frac{h}{2}\, q_{i+1} \tag{7.24c}$$

$$H_i = \frac{h}{2}\,[r_{i+1} + r_i] \tag{7.24d}$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Equation (7.23) gives a system of (N-1) equations with (N+1) unknowns y_o to y_N. The two given boundary conditions (7.18) together with these (N-1) equations are then sufficient to solve for the unknowns y_i's. The solution of the tridiagonal system (7.33) can easily be obtained by using an efficient algorithm called 'Discrete Invariant Imbedding' (Angel and Bellman [4]). In this algorithm we set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \tag{7.25a}$$

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ are to be determined. From (7.25a) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{7.25b}$$

Substituting (7.25b) in (7.23), we get

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}}\, y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \;. \tag{7.25c}$$

By comparing (7.25c) with (7.25a), we get

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \tag{7.26a}$$

$$\text{and} \quad T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}. \tag{7.26b}$$

To solve these recurrence relations for $i = 1, 2, \ldots, N-1$, we need to know the initial conditions for W_0 and T_0. This can be done by considering the boundary condition $y(\delta) = \bar{\alpha}$, as follows

$$y_0 = \bar{\alpha} = W_0 y_1 + T_0$$

If we choose $W_0 = 0$, then $T_0 = \bar{\alpha}$. Using these initial values, we first compute W_i and T_i for $i = 1, 2, \ldots, N-1$ from (7.26a) and (7.26b) in the forward process. Then we obtain the solutions y_i for $i = N-1, N-2, \ldots, 2, 1$; in the backward process from (7.25a) using the remaining boundary condition $y_N = \beta$.

Step 5 : Adjoin the solutions of inner and outer region problems to obtain an approximate solution to the original problem (7.1) - (7.2) over the interval $0 \leq x \leq 1$.

Step 6 : Repeat the process for different choices of x_p (terminal point of the inner region), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|y(x)^{m+1} - y(x)^m| \leq \sigma \; ; \; x_p \leq x \leq 1 \tag{7.27}$$

where

$y(x)^m = m^{th}$ iteration of the outer region solution,

and

σ = prescribed tolerance bound.

Remark 7.2 : We have already mentioned that the differential equation (7.9) is valid over the entire interval $0 \leq t \leq 1/\varepsilon$. Hence, as an alternative to the solution of the outer region problem (7.19) - (7.18), we may use the solution of the initial value problem (7.9) - (7.10) over the interval $x_p \leq x \leq 1$.

7.3 NUMERICAL EXAMPLES

Example 7.1 : Consider the following homogeneous SPP from Kevorkian and Cole [84], Page : 33; Equations : 2.3.26 and 2.3.27 with $\alpha = 0$;

$$\varepsilon y''(x)+y'(x) = 0, \; 0 \leq x \leq 1 \tag{7.28a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{7.28b}$$

The exact solution is given by

$$y(x) = \frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))} . \tag{7.28c}$$

By integrating the equation (7.28a), we get

$$\varepsilon y'(x) + y(x) = K. \tag{7.28d}$$

The constant K is determined by using the equation (7.4), as

$$K = y(1) = 1. \tag{7.28e}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem :

$$Y'(t) + Y(t) = 1 \tag{7.28f}$$

$$\text{for } 0 \leq t \leq t_p \text{ with } Y(0) = 0. \tag{7.28g}$$

The analytical solution of the equation (7.28f), using the initial condition (7.28g), is given by

$$Y(t) = 1-\exp(-t). \tag{7.28h}$$

The terminal boundary condition is obtained from the equation (7.28h) and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{7.28i}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Table 7.1, 7.2, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

Example 7.2 : Consider the following non-homogeneous SPP from fluid dynamics for fluid of small viscosity, Reinhardt [125], Example : 2;

$$\varepsilon y''(x)+y'(x) = 1+2x, \; 0 \leq x \leq 1 \tag{7.29a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{7.29b}$$

The exact solution is given by

$$y(x) = x(x+1-2\varepsilon) + (2\varepsilon-1)\frac{(1-\exp(-x/\varepsilon))}{(1-\exp(-1/\varepsilon))}. \tag{7.29c}$$

By integrating the equation (7.29a), we get

$$\varepsilon y'(x) + y(x) = x + x^2 + K. \tag{7.29d}$$

The constant K is determined by using the equation (7.4), as

$$K = y(1) - 1 - 1^2 = -1. \tag{7.29e}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem :

$$Y'(t) + Y(t) = t\varepsilon + t^2\varepsilon^2 - 1 \tag{7.29f}$$

$$\text{for } 0 \leq t \leq t_p \text{ with } Y(0) = 0. \tag{7.29g}$$

The analytical solution of the equation (7.29f), using the initial condition (7.29g), is given by

$$Y(t) = t\varepsilon+t^2\varepsilon^2-1-\varepsilon(2t\varepsilon+1)+2\varepsilon^2+(1+\varepsilon-2\varepsilon^2)e^{-t}. \tag{7.29h}$$

The terminal boundary condition is obtained from the equation (7.29h), and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{7.29i}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Table 7.3, 7.4, for $\varepsilon = 10^{-3}$, 10^{-4}, respectively.

Example 7.3 : Consider the following SPP with variable coefficients from Kevorkian and Cole [84], Page : 33, Equations : 2.3.26 and 2.3.27 with $\alpha = -\frac{1}{2}$;

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0, \quad 0 \le x \le 1 \tag{7.30a}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \tag{7.30b}$$

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [106], Page : 148; Equation : 4.2.32) as our 'exact' solution

$$y(x) = \frac{1}{2-x} - \frac{1}{2}\exp(-(x - \frac{x^2}{4})/\varepsilon). \tag{7.30c}$$

First we rewrite the equation (7.30a) in the form of equation (7.1), i.e. as

$$\varepsilon y''(x) + [(1 - \frac{x}{2})y(x)]' = 0. \tag{7.30d}$$

By integrating the equation (7.30d), we get

$$\varepsilon y'(x) + (1 - \frac{x}{2})y(x) = K. \tag{7.30e}$$

The constant K is determined by using the equation (7.4), as

$$K = (1 - \frac{1}{2})y(1) = \frac{1}{2}. \tag{7.30f}$$

Then, by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem :

$$Y'(t) + (1 - \frac{t\varepsilon}{2})Y(t) = \frac{1}{2} \tag{7.30g}$$

$$\text{for } 0 \le t \le t_p \text{ with } Y(0) = 0. \tag{7.30h}$$

Since it is difficult to obtain analytical (closed form) solution of the problem (7.30g-h), we have solved it numerically by the classical fourth order Runge-Kutta method, and obtained the terminal boundary condition, $y(x_p) = Y(t_p)$.

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Table 7.5, 7.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

7.4 A MORE GENERAL CLASS OF PROBLEMS

In this section, we extend our method to a more general class of problems of the form :

$$\varepsilon y''(x)+[a(x)y(x)]' + b(x)y(x) = h(x), 0 \leq x \leq 1 \qquad (7.31)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta, \qquad (7.32)$$

where ε is a small positive parameter; α, β are given constants; $a(x)$, $b(x)$ and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$; and $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant.

It is clear that there will be a difficulty in applying Step 2 (i.e. in the integration process) due to the presence of the term $b(x)y(x)$. To overcome this difficulty, we first modify the equation (7.31) and then apply our method. For convenience, we shall term this extra step as the 'preliminary step'.

Let y_0 be the solution of the reduced problem of (7.31) - (7.32); that is

$$[a(x)\ y_0(x)]' + b(x)y_0(x) = h(x) \qquad (7.33a)$$

$$\text{with } y_0(1) = \beta. \qquad (7.33b)$$

Preliminary step : Set up the approximate equation to the given equation (7.31) as follows :

$$\varepsilon y''(x)+[a(x)y(x)]' + b(x)y_o(x) = h(x) \qquad (7.34a)$$

where we have simply replaced by y(x) - term by $y_o(x)$, the solution of the reduced problem (7.33a-b). Then, we rewrite this equation(7.34a) in the form of the equation (7.1), i.e. as

$$\varepsilon y''(x) + [a(x)y(x)]' = H(x) \qquad (7.34b)$$

where $H(x) = h(x)-b(x)y_o(x)$.

Now, we can apply our method, Step 1 to Step 6, to the modified problem (7.34b) - (7.32).

In order to verify this approach, we discuss one simple example in detail.

Example 7.4 : Consider the following singular perturbation problem from Bender and Orszag [22], Page : 480; Problem : 9.17 with $\alpha = 0$;

$$\varepsilon y''(x) + y'(x) - y(x) = 0, \; 0 \leq x \leq 1 \qquad (7.35a)$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 1. \qquad (7.35b)$$

The exact solution is given by

$$y(x) = \frac{(e^{m_2}-1)e^{m_1x} + (1-e^{m_1})e^{m_2x}}{(e^{m_2} - e^{m_1})} \qquad (7.35c)$$

where $m_1 = \dfrac{-1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$ and $m_2 = \dfrac{-1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$.

From the Preliminary step, we get an approximate equation to (7.35a) as

$$\varepsilon y''(x) + y'(x) - y_o(x) = 0 \tag{7.35d}$$

where $y_o(x) = e^{x-1}$ is the solution of the reduced problem of (7.35a-b), that is

$$y_o'(x) - y_o(x) = 0 \tag{7.35e}$$

$$\text{with } y_o(1) = 1. \tag{7.35f}$$

Then rewrite the equation (7.35d) in the form of (7.1) :

$$\varepsilon y''(x) + y'(x) = e^{x-1}. \tag{7.35g}$$

By integrating the equation (7.35g), we get

$$\varepsilon y'(x) + y(x) = e^{x-1} + K. \tag{7.35h}$$

The constant K is determined by using the equation (7.4), as

$$K = y(1) - e^{1-1} = 0. \tag{7.35i}$$

Then by using the scaling $t = x/\varepsilon$, we get the inner region problem as an initial value problem :

$$Y'(t) + Y(t) = e^{t\varepsilon-1}, \tag{7.35j}$$

$$\text{for } 0 \leq t \leq t_p \text{ with } Y(0) = 1. \tag{7.35k}$$

The analytical solution of the equation (7.35j), using the initial condition (7.35k), is given by

$$Y(t) = \frac{e^{t\varepsilon-1}}{(1+\varepsilon)} + \left[1 - \frac{e^{-1}}{(1+\varepsilon)}\right] e^{-t}. \tag{7.35l}$$

The terminal boundary condition is obtained from the equation (7.351) and denoted by

$$y(x_p) = Y(t_p) = \bar{\alpha}. \tag{7.35m}$$

In turn, the outer region problem is obtained and solved by using Step 4.

The computational results are presented in the Table 7.7, 7.8, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

7.5 DISCUSSION AND CONCLUSIONS

We have described an approximate method for the numerical solution of a class of singular perturbation problems. As mentioned the method is iterative on the terminal point of the inner region. The process is to be repeated for various choices of x_p (terminal point of the inner region), until the solution profiles stabilize in both the regions. As an alternative of the solution of the outer region problem (7.19) - (7.18), we may use the solution of the initial value problem (7.9) - (7.10) over the interval $x_p \leq x \leq 1$. But for better accurate results, we prefer to solve the outer region problem (7.19)-(7.18) as it is. We have implemented this method on three problems, by taking different values of ε. Only one extra step, called the 'Preliminary step', is needed to apply the present method to a more general class of problems. We have verified this by solving a more general class of problem. It can be observed from the tables that the present method approximates the exact solution very well.

Table 7.1

Computational results for Example 7.1, $\varepsilon = 10^{-3}$

t_p → x	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.39346933	0.39346933	0.39346933
$1.0(10^{-3})$	0.63212056	0.63212056	0.63212056
$2.5(10^{-3})$	0.91791500	0.91791500	0.91791500
$5.0(10^{-3})$	<u>0.99326205</u>	0.99326205	0.99326205
$7.5(10^{-3})$		0.99944691	0.99944691
$1.0(10^{-2})$		<u>0.99995460</u>	0.99995460
$2.0(10^{-2})$			1.00000000
0.1	1.00000000	1.00000000	1.00000000
0.2	1.00000000	1.00000000	1.00000000
0.3	1.00000000	1.00000000	1.00000000
0.4	1.00000000	1.00000000	1.00000000
0.5	1.00000000	1.00000000	1.00000000
0.6	1.00000000	1.00000000	1.00000000
0.7	1.00000000	1.00000000	1.00000000
0.8	1.00000000	1.00000000	1.00000000
0.9	1.00000000	1.00000000	1.00000000
1.0	1.00000000	1.00000000	1.00000000

Table 7.2

Computational results for Example 7.1 , $\varepsilon = 10^{-4}$

t_p → x	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.39346933	0.39346933	0.39346933
$1.0(10^{-4})$	0.63212056	0.63212056	0.63212056
$2.5(10^{-4})$	0.91791500	0.91791500	0.91791500
$5.0(10^{-4})$	<u>0.99326205</u>	0.99326205	0.99326205
$7.5(10^{-4})$		0.99944691	0.99944691
$1.0(10^{-3})$		<u>0.99995460</u>	0.99995460
$2.0(10^{-3})$			1.00000000
0.1	1.00000000	1.00000000	1.00000000
0.2	1.00000000	1.00000000	1.00000000
0.3	1.00000000	1.00000000	1.00000000
0.4	1.00000000	1.00000000	1.00000000
0.5	1.00000000	1.00000000	1.00000000
0.6	1.00000000	1.00000000	1.00000000
0.7	1.00000000	1.00000000	1.00000000
0.8	1.00000000	1.00000000	1.00000000
0.9	1.00000000	1.00000000	1.00000000
1.0	1.00000000	1.00000000	1.00000000

Table 7.3

Computational results for Example 7.2, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	-0.39336275	-0.39336275	-0.39218315
$1.0(10^{-3})$	-0.63175242	-0.63175242	-0.62985732
$2.5(10^{-3})$	-0.91632984	-0.91632984	-0.91357792
$5.0(10^{-3})$	<u>-0.98923834</u>	-0.98923834	-0.98626053
$7.5(10^{-3})$		-0.99290311	-0.98990677
$1.0(10^{-2})$		<u>-0.99087255</u>	-0.98787469
$2.0(10^{-2})$			-0.97764000
0.1	-0.88964006	-0.88820001	-0.88820001
0.2	-0.75968005	-0.75840001	-0.75840001
0.3	-0.60972004	-0.60860001	-0.60860001
0.4	-0.43976007	-0.43880000	-0.43880001
0.5	-0.24980013	-0.24899999	-0.24900001
0.6	-0.03984018	-0.03919998	-0.03920000
0.7	0.19011980	0.19060002	0.19060000
0.8	0.44007983	0.44040002	0.44039998
0.9	0.71003990	0.71020001	0.71019998
1.0	1.00000000	1.00000000	1.00000000

Table 7.4

Computational results for Example 7.2, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ / x	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	-0.39345868	-0.39345868	-0.39334064
$1.0(10^{-4})$	-0.63208377	-0.63208377	-0.63189415
$2.5(10^{-4})$	-0.91775677	-0.91775677	-0.91748140
$5.0(10^{-4})$	-0.99286120	-0.99286120	-0.99256325
$7.5(10^{-4})$		-0.99879643	-0.99849661
$1.0(10^{-3})$		-0.99905378	-0.99875381
$2.0(10^{-3})$			-0.99779639
0.1	-0.88997293	-0.88982004	-0.88982000
0.2	-0.75997696	-0.75984003	-0.75984000
0.3	-0.60998099	-0.60986003	-0.60986000
0.4	-0.43998471	-0.43988004	-0.43988000
0.5	-0.24998811	-0.24990011	-0.24990000
0.6	-0.03999116	-0.03992017	-0.03991999
0.7	0.19000613	0.19005981	0.19006000
0.8	0.44000375	0.44003983	0.44003999
0.9	0.71000171	0.71001990	0.71002001
1.0	1.00000000	1.00000000	1.00000000

Table 7.5

Computational results for Example 7.3, $\varepsilon = 10^{-3}$

t_p → / x	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-4})$	0.19674221	0.19674221	0.19684075
$1.0(10^{-3})$	0.31610610	0.31610610	0.31626441
$2.5(10^{-3})$	0.45928897	0.45928897	0.45951910
$5.0(10^{-3})$	<u>0.49761321</u>	0.49761321	0.49786303
$7.5(10^{-3})$		0.50134929	0.50160159
$1.0(10^{-2})$		<u>0.50223590</u>	0.50248929
$2.0(10^{-2})$			0.50505050
0.1	0.52646507	0.52705440	0.52631579
0.2	0.55570632	0.55630083	0.55555556
0.3	0.58838637	0.58898142	0.58823530
0.4	0.62514970	0.62573835	0.62500000
0.5	0.66681240	0.66738427	0.66666666
0.6	0.71442340	0.71496310	0.71428571
0.7	0.76935448	0.76983835	0.76923077
0.8	0.83343384	0.83382540	0.83333333
0.9	0.90915328	0.90939540	0.90909090
1.0	1.00000000	1.00000000	1.00000000

Table 7.6

Computational results for Example 7.3, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	Exact solution
0.0	0.00000000	0.00000000	0.00000000
$5.0(10^{-5})$	0.19673530	0.19673530	0.19674528
$1.0(10^{-4})$	0.31606472	0.31606472	0.31608069
$2.5(10^{-4})$	0.45899055	0.45899055	0.45901360
$5.0(10^{-4})$	<u>0.49672909</u>	0.49672909	0.49675395
$7.5(10^{-4})$		0.49988562	0.49991064
$1.0(10^{-3})$		<u>0.50020234</u>	0.50022737
$2.0(10^{-3})$			0.50050050
0.1	0.52632324	0.52639090	0.52631579
0.2	0.55556350	0.55563144	0.55555556
0.3	0.58824352	0.58831131	0.58823530
0.4	0.62500860	0.62507524	0.62500000
0.5	0.66667552	0.66673988	0.66666666
0.6	0.71429463	0.71435488	0.71428571
0.7	0.76923909	0.76929286	0.76923077
0.8	0.83334073	0.83338355	0.83333333
0.9	0.90909586	0.90912206	0.90909090
1.0	1.00000000	1.00000000	1.00000000

Table 7.7

Computational results for Example 7.4, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Exact solution
0.0	1.00000000	1.00000000	1.00000000
$5.0(10^{-4})$	0.75131915	0.75131915	0.75141688
$1.0(10^{-3})$	0.60055898	0.60055898	0.60079141
$2.5(10^{-3})$	0.42034964	0.42034964	0.42089517
$5.0(10^{-3})$	0.37361576	0.37361576	0.37432568
$7.5(10^{-3})$		0.37062845	0.37136243
$1.0(10^{-2})$		0.37123420	0.37197179
$2.0(10^{-2})$			0.37567774
0.1	0.40663699	0.40693049	0.40693440
0.2	0.44939510	0.44968342	0.44968726
0.3	0.49664925	0.49692805	0.49693177
0.4	0.54887223	0.54913630	0.54913982
0.5	0.60658645	0.60682965	0.60683289
0.6	0.67036937	0.67058437	0.67058726
0.7	0.74085910	0.74103729	0.74103969
0.8	0.81876083	0.81889214	0.81889392
0.9	0.90485402	0.90492660	0.90492758
1.0	1.00000000	1.00000000	1.00000000

Table 7.8

Computational results for Example 7.4, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	5 y(x)	10 y(x)	Exact solution
0.0	1.00000000	1.00000000	1.00000000
$5.0(10^{-5})$	0.75128387	0.75128387	0.75128750
$1.0(10^{-4})$	0.60043713	0.60043713	0.60045052
$2.5(10^{-4})$	0.41982526	0.41982526	0.41986551
$5.0(10^{-4})$	0.37228607	0.37228607	0.37234153
$7.5(10^{-4})$		0.36846828	0.36852598
$1.0(10^{-3})$		0.36823938	0.36829735
$2.0(10^{-3})$			0.36863712
0.1	0.40651796	0.40660421	0.40659073
0.2	0.44927818	0.44936290	0.44934966
0.3	0.49653622	0.49661812	0.49660532
0.4	0.54876515	0.54884274	0.54883059
0.5	0.60648782	0.60655932	0.60654812
0.6	0.67028214	0.67034537	0.67033548
0.7	0.74078687	0.74083919	0.74083101
0.8	0.81870759	0.81874620	0.81874018
0.9	0.90482465	0.90484595	0.90484262
1.0	1.00000000	1.00000000	1.00000000

CHAPTER 8

AN INITIAL VALUE TECHNIQUE FOR A CLASS OF NONLINEAR SINGULAR PERTURBATION PROBLEMS

8.1 INTRODUCTION

In this and next chapter we discuss numerical techniques for solving nonlinear singularly perturbed two point boundary value problems. There are a wide variety of techniques for the solution of nonlinear singular perturbation problems (cf. Bender and Orszag [22], Kevorkian and Cole [84], and O'Malley [113]). Many of these techniques consists of (a) dividing the problem into an inner region problem and an outer region problem, (b) expressing the inner and outer solutions as asymptotic expansions, (c) equating various terms in the inner and outer expansions to determine the constants in these expansions, and (d) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. Typically the inner region problems are obtained from the original problem by rescaling the independent variable. These techniques and their variations have been used successfully on a variety of linear and nonlinear singular perturbation problems. However, there can be difficulties in applying these methods, such as the matching of the coefficients of the inner and outer expansions. Success may depend on finding the proper scaling to express the dependent and independent variables.

In view of the wealth of literature on singular perturbation problems, we raise the question whether there are other ways to attack singular perturbation problems; ways that are very easy to use, and ready for computer implementation. In response to this need for a fresh approach to singular perturbation problems, we propose and illustrate in this chapter the initial value technique for a class of nonlinear singularly perturbed two point boundary value problems with a boundary layer on the left end of the underlying interval. It is distinguished by the following fact: The original second order problem is replaced by an asymptotically equivalent first order problem and solved as an initial value problem. Numerical experience with several examples is described.

8.2 INITIAL VALUE TECHNIQUE

For convenience we call our method the 'initial value technique'. To set the stage for the initial value technique, we consider a class of nonlinear singular perturbation problems of the form :

$$\varepsilon y''(x) + [p(y(x))]' + q(x,y(x)) = r(x), a \leq x \leq b \quad (8.1)$$

$$\text{with } y(a) = \alpha \text{ and } y(b) = \beta \quad (8.2)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $p(y)$, $q(x,y)$ and $r(x)$ are assumed to be sufficiently differentiable functions. Furthermore, we assume that the problem (8.1) - (8.2) has a solution which displays a boundary layer of width $O(\varepsilon)$ at $x = a$ for small values of ε.

The initial value method consists of the following steps:

Step 1 : Obtain the reduced problem by setting ε equal to zero in (8.1), and solve it for the solution. Let $y_0(x)$ be the solution of the reduced problem of (8.1) - (8.2); that is

$$[p(y_0(x))]' + q(x, y_0(x)) = r(x) \tag{8.3}$$

$$\text{with } y_0(b) = \beta \; . \tag{8.4}$$

Step 2 : Set up the approximate equation to the given equation (8.1) as follows :

$$\varepsilon y''(x) + [p(y(x))]' + q(x, y_0(x)) = r(x) \tag{8.5}$$

where we have simply replaced the $y(x)$-term in $q(x, y(x))$ by $y_0(x)$, the solution of the reduced problem (8.3) - (8.4).

Step 3 : Replace the approximated second order problem (8.5) - (8.2) by an asymptotically equivalent first order problem as follows :

By integrating the equation (8.5), we obtain

$$\varepsilon y'(x) + p(y(x)) + Q(x) = R(x) + K \tag{8.6}$$

where we have set

$$Q(x) = \int q(x, y_0(x))\, dx,$$

$$R(x) = \int r(x)\, dx$$

and K is a constant to be determined.

In order to determine the constant K, we introduce the condition that the reduced equation of the equation (8.6) should satisfy the boundary condition, $y(b) = \beta$.

$$\text{i.e.}\quad p(y(b)) + Q(b) = R(b) + K$$

$$\therefore \quad K = p(\beta) + Q(b) - R(b). \tag{8.7}$$

Remark 8.1 : This choice of K ensures that the solution of the reduced problem of (8.1) - (8.2) satisfies the reduced equation of the equation (8.6).

Now, we adjoin the condition (which we drop, whenever we formulate the reduced problem of (8.1) - (8.2)) $y(a) = \alpha$ to the equation (8.6) to obtain a first order problem as follows :

$$\varepsilon y'(x) + p(y(x)) + Q(x) = R(x) + K \tag{8.8}$$

$$\text{with } y(a) = \alpha \tag{8.9}$$

where the constant K is given by the equation (8.7). Thus in a manner of speaking, we have replaced the original second order problem (8.1) - (8.2) with the asymptotically equivalent first order problem (8.8) - (8.9). We solve this initial value problem (8.8) - (8.9) to obtain the solutions over the interval $a \leq x \leq b$. There now exists a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation, we make use of the classical fourth order Runge-Kutta method. In fact any standard analytical or numerical method can be used.

Remark 8.2 : For the case $q(x, y(x)) = 0$, we do not require the Steps 1 and 2, because we can directly integrate the given equation.

8.3 NUMERICAL EXAMPLES

To illustrate how the initial value method works we will discuss three examples solved by the method. These examples have been chosen because they have been widely discussed in the literature and because approximate solutions are available for comparison.

Example 8.1 : Consider the following example from O'Malley [113], Page : 9; Equation : 1.10; Case 2;

$$\varepsilon y'' = yy' \; ; \; -1 \leq x \leq 1 \tag{8.10}$$

$$\text{with } y(-1) = 0 \text{ and } y(1) = -1. \tag{8.11}$$

We have chosen to use O'Malley's approximate solution (O'Malley [113], Pages : 9 and 10; Equations : 1.13 and 1.14) for comparison.

$$y(x) = -\frac{[1-\exp(-(x+1)/\varepsilon)]}{[1+\exp(-(x+1)/\varepsilon)]} \, . \tag{8.12}$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = -1$ (cf. O'Malley [113], Pages : 9 and 10; Equations : 1.10, 1.13 and 1.14; Case 2, and Roberts [129]).

First we rewrite the equation (8.10) in the form of the equation (8.1), i.e. as

$$\varepsilon y''(x) - [\frac{y(x)^2}{2}]' = 0. \tag{8.13}$$

As mentioned in the Remark 8.2 we do not require the Steps 1 and 2 since the term $q(x,y(x)) = 0$ in this example. Hence by integrating the equation (8.13), we get

$$\varepsilon y'(x) - \frac{y(x)^2}{2} = K. \tag{8.14}$$

The constant K is determined by using the equation (8.7), as

$$K = -\frac{y(1)^2}{2} = -\frac{1}{2}. \tag{8.15}$$

Hence the initial value problem is given by

$$\varepsilon y'(x) - \left[\frac{y(x)^2 - 1}{2}\right] = 0 \tag{8.16}$$

$$\text{for } -1 \leq x \leq 1 \text{ with } y(-1) = 0. \tag{8.17}$$

The computational results are presented in the Table 8.1, 8.2, for $\varepsilon = 10^{-2}, 10^{-3}$, respectively.

Example 8.2 : Consider the following example from Kevorkian and Cole [84], Page : 56; Equations : 2.5.1;

$$\varepsilon y'' + yy' - y = 0; \ 0 \leq x \leq 1 \tag{8.18}$$

$$\text{with } y(0) = -1 \text{ and } y(1) = 3.9995. \tag{8.19}$$

We have chosen to use Kevorkian and Cole's uniformly valid approximation (Kevorkian and Cole [84], Pages : 57 and 58; Equations : 2.5.5, 2.5.11 and 2.5.14) for comparison.

$$y(x) = x + c_1 \tanh \frac{c_1}{2}\left(\frac{x}{\varepsilon} + c_2\right) \tag{8.20}$$

where $c_1 = 2.9995$

and $c_2 = \dfrac{1}{c_1} \log \left(\dfrac{c_1 - 1}{c_1 + 1}\right)$.

For this example, we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Kevorkian and Cole [84], Pages : 56-66, and Roberts [130]). First we write the equation (8.18) in the form of the equation (8.1), i.e. as

$$\varepsilon y''(x) + [\frac{y(x)^2}{2}]' - y(x) = 0. \tag{8.21}$$

The reduced equation is given by

$$[\frac{y(x)^2}{2}]' - y(x) = 0. \tag{8.22}$$

This implies two equations

$$y(x) = 0, \tag{8.23}$$

$$y'(x) = 1. \tag{8.24}$$

Only the equation (8.24) is the correct differential equation (cf. Kevorkian and Cole [84] and Roberts [130]). Hence the 'correct' reduced problem is given by

$$y_o'(x) = 1 \tag{8.25}$$

$$\text{with } y_o(1) = 3.9995 \tag{8.26}$$

whose solution is

$$y_o(x) = x+2.9995. \tag{8.27}$$

From the Step 2, we get an approximate equation to (8.21) as

$$\varepsilon y''(x) + [\frac{y(x)^2}{2}]' - (x+2.9995) = 0. \tag{8.28}$$

Now by integrating the equation (8.28), we get

$$\varepsilon y'(x) + \frac{y(x)^2}{2} - (\frac{x^2}{2} + (\beta-1)x) = K \tag{8.29}$$

where we have set $\beta = 3.9995$.

The constant K is determined by using the equation (8.7), as

$$K = \frac{y(1)^2}{2} - (\frac{1}{2} + \beta-1) = \frac{(\beta-1)^2}{2}. \tag{8.30}$$

Hence the initial value problem is given by

$$\varepsilon y'(x) + \left[\frac{y(x)^2 - (x+\beta-1)^2}{2}\right] = 0 \tag{8.31}$$

$$\text{for } 0 \leq x \leq 1 \text{ with } y(0) = -1. \tag{8.32}$$

The computational results are presented in the Table 8.3, 8.4, for $\varepsilon = 10^{-2}$, 10^{-3}, respectively.

Example 8.3 : Consider the following example from Bender and Orszag [22], Page : 463; Equations: 9.7.1;

$$\varepsilon y'' + 2y' + e^{y} = 0; \ 0 \leq x \leq 1 \tag{8.33}$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 0. \tag{8.34}$$

We have chosen to use Bender and Orszag's uniformly valid approximation (Bender and Orszag [22], Page : 463; Equation : 9.7.6) for comparison.

$$y(x) = \log \frac{2}{1+x} - (\exp(-2x/\varepsilon)) \log 2 \tag{8.35}$$

For this example also we have a boundary layer at $x = 0$ (cf. Bender and Orszag [22]).

The reduced problem is given by

$$2y_0'(x) + e^{y_0(x)} = 0, \tag{8.36}$$

$$\text{with } y_0(1) = 0, \tag{8.37}$$

whose solution is

$$y_0(x) = \log \frac{2}{1+x}. \tag{8.38}$$

Using the Step 2, we get an approximate equation to (8.33) as

$$\varepsilon y''(x) + [2y(x)]' + \frac{2}{1+x} = 0. \tag{8.39}$$

Now by integrating the equation (8.39), we get

$$\varepsilon y'(x) + 2y(x) + 2 \log (1+x) = K. \tag{8.40}$$

The constant K is determined by using the equation (8.7), as

$$K = 2y(1) + 2 \log (1+1) = 2 \log 2. \tag{8.41}$$

Hence the initial value problem is given by

$$\varepsilon y'(x) + 2 \left[y(x) - \log \frac{2}{1+x}\right] = 0 \tag{8.42}$$

$$\text{for } 0 \leq x \leq 1 \text{ with } y(0) = 0. \tag{8.43}$$

The computational results are presented in the Table 8.5, 8.6, for $\varepsilon = 10^{-2}, 10^{-3}$, respectively.

8.4 DISCUSSION

In general the numerical solution of a boundary value problem will be a more difficult matter than the numerical solution of the corresponding initial value problem. Hence, we always prefer to convert the second order problem into a first order problem. For example, we have a transformation which converts a second order linear differential equation into a first order nonlinear differential equation, called the Riccati equation. If the original problem is nonlinear then we have to solve by iterative methods which require more computational time. In this situation the initial value technique is required. On applying the initial value technique, the expectation is that solving the corresponding initial value problem will be easier

8.5 CONCLUSIONS

We have described and illustrated with three examples the initial value technique. It provides an alternative and supplementary technique to the conventional ways of solving certain classes of nonlinear singular perturbation problems. The method possesses several advantages. First, it does not require the analysis, experimentation, and knowledge necessary in the conventional methods to find the 'correct' asymptotic expansion. Second, the method is primarily a numerical technique and is readily adapted for computer implementation with a modest amount of problem preparation. Third, it is accurate and efficient. The numerical experiments illustrate this fact. Finally, we conclude that the present method appears to be one of the best choices for numerically solving singular perturbation problems with less amount of computational time.

Table 8.1

Computational results for Example 8.1, $\varepsilon = 10^{-2}$

x	Present solution y(x)	O'Malley's solution [113] y(x)
-1.000	0.0000000	0.0000000
-0.999	-0.0499584	-0.0499585
-0.995	-0.2449187	-0.2449188
-0.990	-0.4621171	-0.4621171
-0.985	-0.6351489	-0.6351490
-0.980	-0.7615941	-0.7615942
-0.975	-0.8482836	-0.8482836
-0.970	-0.9051482	-0.9051483
-0.965	-0.9413755	-0.9413755
-0.960	-0.9640275	-0.9640276
-0.950	-0.9866143	-0.9866143
-0.8	-1.0000000	-1.0000000
-0.6	-1.0000000	-1.0000000
-0.4	-1.0000000	-1.0000000
-0.2	-1.0000000	-1.0000000
0.0	-1.0000000	-1.0000000
0.2	-1.0000000	-1.0000000
0.4	-1.0000000	-1.0000000
0.6	-1.0000000	-1.0000000
0.8	-1.0000000	-1.0000000
1.0	-1.0000000	-1.0000000

Table 8.2

Computational results for Example 8.1, $\varepsilon=10^{-3}$

x	Present solution y(x)	O'Malley's solution [113] y(x)
-1.0000	0.0000000	0.0000000
-0.9999	-0.0499584	-0.0499592
-0.9995	-0.2449187	-0.2449191
-0.9990	-0.4621171	-0.4621180
-0.9985	-0.6351489	-0.6351499
-0.9980	-0.7615941	-0.7615934
-0.9975	-0.8482836	-0.8482833
-0.9970	-0.9051482	-0.9051481
-0.9965	-0.9413755	-0.9413755
-0.9960	-0.9640275	-0.9640276
-0.9500	-1.0000000	-1.0000000
-0.8	-1.0000000	-1.0000000
-0.6	-1.0000000	-1.0000000
-0.4	-1.0000000	-1.0000000
-0.2	-1.0000000	-1.0000000
0.0	-1.0000000	-1.0000000
0.2	-1.0000000	-1.0000000
0.4	-1.0000000	-1.0000000
0.6	-1.0000000	-1.0000000
0.8	-1.0000000	-1.0000000
1.0	-1.0000000	-1.0000000

Table 8.3

Computational results for Example 8.2, $\varepsilon = 10^{-2}$

x	Present solution $y(x)$	Kevorkian and Cole's solution [84] $y(x)$
0.0	-1.0000000	-1.0000000
$1.0(10^{-3})$	-0.5822459	-0.5813961
$5.0(10^{-3})$	1.1513299	1.1529592
$1.0(10^{-2})$	2.4644777	2.4659396
$1.5(10^{-2})$	2.8817559	2.8839996
$2.0(10^{-2})$	2.9870105	2.9898735
$2.5(10^{-2})$	3.0147144	3.0178623
$3.0(10^{-2})$	3.0247648	3.0280173
$3.5(10^{-2})$	3.0308848	3.0341690
$4.0(10^{-2})$	3.0361357	3.0394261
$5.0(10^{-2})$	3.0462122	3.0494963
0.1	3.0962686	3.0995000
0.2	3.1963699	3.1995000
0.3	3.2964650	3.2995000
0.4	3.3965544	3.3995000
0.5	3.4466386	3.4995000
0.6	3.5967184	3.5995000
0.7	3.6967939	3.6995000
0.8	3.7968656	3.7995000
0.9	3.8969335	3.8995000
1.0	3.9969980	3.9995000

Table 8.4

Computational results for Example 8.2, $\varepsilon = 10^{-3}$

x	Present solution $y(x)$	Kevorkian and Cole's solution [84] $y(x)$
0.0	-1.0000000	-1.0000000
$1.0(10^{-4})$	-0.5823840	-0.5822961
$5.0(10^{-4})$	1.1482856	1.1484592
$1.0(10^{-3})$	2.4567688	2.4569396
$1.5(10^{-3})$	2.8702568	2.8704996
$2.0(10^{-3})$	2.9715787	2.9718735
$2.5(10^{-3})$	2.9950433	2.9953623
$3.0(10^{-3})$	3.0006889	3.0010173
$3.5(10^{-3})$	3.0023375	3.0026690
$4.0(10^{-3})$	3.0030936	3.0034261
$5.0(10^{-2})$	3.0491720	3.0495000
0.1	3.0991772	3.0995000
0.2	3.1991872	3.1995000
0.3	3.2991974	3.2995000
0.4	3.3992080	3.3995000
0.5	3.4992181	3.4995000
0.6	3.5992277	3.5995000
0.7	3.6992369	3.6995000
0.8	3.7992457	3.7995000
0.9	3.8992542	3.8995000
1.0	3.9992623	3.9995000

Table 8.5

Computational results for Example 8.3, $\varepsilon = 10^{-2}$

x	Present solution y(x)	Bender and Orszag's solution [22] y(x)
0.0	0.0000000	0.0000000
$1.0(10^{-3})$	0.1255508	0.1246468
$5.0(10^{-3})$	0.4363124	0.4331651
$1.0(10^{-2})$	0.5936818	0.5893896
$1.5(10^{-2})$	0.6484476	0.6437488
$2.0(10^{-2})$	0.6654825	0.6606491
$2.5(10^{-2})$	0.6686520	0.6637842
$3.0(10^{-2})$	0.6667358	0.6618702
$3.5(10^{-2})$	0.6629635	0.6581137
$4.0(10^{-2})$	0.6585233	0.6536939
$5.0(10^{-2})$	0.6491101	0.6443255
0.1	0.6024033	0.5973370
0.2	0.5150098	0.5108256
0.3	0.4346440	0.4307829
0.4	0.3602594	0.3566749
0.5	0.2910268	0.2876821
0.6	0.2262785	0.2231435
0.7	0.1654688	0.1625189
0.8	0.1081459	0.1053605
0.9	0.0539316	0.0512933
1.0	0.0025060	0.0000000

Table 8.6

Computational results for Example 8.3, $\varepsilon = 10^{-3}$

x	Present solution $y(x)$	Bender and Orszag's solution [22] $y(x)$
0.0	0.0000000	0.0000000
$1.0(10^{-4})$	0.1256351	0.1255463
$5.0(10^{-4})$	0.4379647	0.4376527
$1.0(10^{-3})$	0.5987695	0.5983404
$1.5(10^{-3})$	0.6576115	0.6571385
$2.0(10^{-3})$	0.6789430	0.6784537
$2.5(10^{-3})$	0.6864752	0.6859799
$3.0(10^{-3})$	0.6889309	0.6884335
$3.5(10^{-3})$	0.6895192	0.6890212
$4.0(10^{-3})$	0.6894207	0.6889226
$5.0(10^{-2})$	0.6448335	0.6443570
0.1	0.5982919	0.5978370
0.2	0.5112427	0.5108256
0.3	0.4311673	0.4307829
0.4	0.3570307	0.3566749
0.5	0.2880129	0.2876821
0.6	0.2234527	0.2231435
0.7	0.1628089	0.1625189
0.8	0.1056334	0.1053605
0.9	0.0515509	0.0512933
1.0	0.0002439	0.0000000

CHAPTER 9

A BOUNDARY VALUE METHOD FOR A CLASS OF NONLINEAR SINGULAR PERTURBATION PROBLEMS

9.1 INTRODUCTION

In order to know the behavior of the solution of the singular perturbation problem in the boundary layer region it is always suggestive to solve the problem in outer and boundary layer regions separately. For many singular perturbation problems a reduced problem is well defined and known a priori. We use these facts for the numerical solution of a class of nonlinear singular perturbation problems.

In this chapter, we propose and illustrate a boundary value method for solving a class of nonlinear singularly perturbed two point boundary value problems with a boundary layer on the left end of the underlying interval. This method is based on a different approach which is conceptually closer to the ideas of singular perturbation analysis. By constructing a modified problem with a boundary layer correction, we discuss how to deal with the boundary layer separately. The method is iterative on the terminal point of the boundary layer. Several numerical examples are discussed to illustrate the method. Finally, a lower bound for the terminal point of the boundary layer region is obtained in terms of the perturbation parameter.

9.2 BOUNDARY VALUE METHOD

For convenience we call our method the 'boundary value method'. For clarity of exposition of the boundary value method, we consider a class of nonlinear singular perturbation problems of the form :

$$\varepsilon y'' + f(x,y)y' + g(x,y) = 0, \quad 0 \leq x \leq 1 \tag{9.1}$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta, \tag{9.2}$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $f(x,y)$, and $g(x,y)$ are assumed to be sufficiently differentiable functions. Furthermore, we assume that the problem (9.1) - (9.2) has a solution which displays a boundary layer of width $O(\varepsilon)$ at $x = 0$ for small values of ε.

We are interested in solving the problem (9.1) - (9.2) numerically for small values of the perturbation parameter ε. Upon setting $\varepsilon = 0$ in (9.1) we note that the equation decreases in order, that is, it becomes

$$f(x,y)y' + g(x,y) = 0. \tag{9.3}$$

In order to determine the solution of this equation we only need to impose one of the boundary conditions from (9.2). The question is which one of the two conditions should be retained. The answer depends upon the location of the boundary layer.

Since we have assumed that the boundary layer occurs in the neighbourhood of $x = 0$, we must drop the boundary condition at the origin and take the boundary condition at the other end,

that is, $y(1) = \beta$ should be retained. Hence the equation (9.3) together with the boundary condition at $x = 1$ results in the so-called reduced problem :

$$f(x,U)U' + g(x,U) = 0, \tag{9.4}$$

$$U(1) = \beta. \tag{9.5}$$

It is well known from the singular perturbation theory that over most of the interval $[0,1]$ the solution of (9.1) - (9.2) behaves like the solution of (9.4) - (9.5) but to satisfy the other boundary condition there is a small region in which the solution of (9.1) - (9.2) must deviate greatly from that of (9.4) - (9.5). This region is usually referred to as the boundary layer region. We choose the so-called stretching transformation :

$$x = t\varepsilon. \tag{9.6}$$

This transforms the equation (9.1) into

$$\frac{d^2y}{dt^2} + f(t\varepsilon,y)\frac{dy}{dt} + \varepsilon g(t\varepsilon,y) = 0. \tag{9.7}$$

Upon setting $\varepsilon = 0$ in (9.7), we have

$$\frac{d^2y}{dt^2} + f(0,y)\frac{dy}{dt} = 0. \tag{9.8}$$

If we require the solution of (9.8) to compensate for the fact that the solution of the reduced problem (9.4) - (9.5) does not satisfy the boundary condition at $x = 0$ and further require that this solution goes to zero as $t \to +\infty$ then we obtain the boundary layer correction problem :

$$V'' + f(0,U(0)+V)V' = 0, \ t > 0, \tag{9.9}$$

$$V(0) = \alpha-U(0), \ \lim_{t\to+\infty} V(t) = 0. \tag{9.10}$$

Then from the standard singular perturbation theory (see e.g. O'Malley [113]) it follows that the solution of (9.1) - (9.2) admits the representation in terms of the solutions of the reduced and boundary layer correction problems. Thus one can write the solution of (9.1) - (9.2) as an asymptotic expansion

$$y(x) = U(x) + V(t) + O(\varepsilon) \tag{9.11}$$

as $\varepsilon \to 0^+$ uniformly in $[0,1]$, with U the solution of (9.4)-(9.5) and V the solution of (9.9) - (9.10).

The expansion (9.11) will be the basis for our numerical scheme. We note that (9.9) - (9.10) is an exterior boundary value problem in terms of the stretched variable t. Hence the usual numerical methods can not be directly applied to (9.9) - (9.10) without some modification. As will be seen, our boundary layer correction problem is one such modification.

The idea of our scheme is to construct U and V in (9.11) so that the solution

$$y = U+V \tag{9.12}$$

can be used to approximate the solution y(x) of (9.1) - (9.2). In the case of the reduced problem (9.4) - (9.5), since no small parameter ε is involved, we can approximate U without difficulty by any suitable method. Although ε does not appear explicitly in the boundary layer correction problem(9.9) - (9.10), the

semi-infinite domain causes some difficulty. Hence our first task is to modify (9.9) - (9.10) in such a way that we treat a similar problem but in a finite interval which will enable us to utilize standard numerical methods for boundary value problems. Let t_p be the terminal point or width or thickness of the boundary layer region. Then from the theory of singular perturbations, we know that V the solution of the boundary layer correction problem is insignificant outside the boundary layer region. Hence, in place of the boundary layer correction problem (9.9) - (9.10), we consider the following modified boundary layer correction problem :

$$V'' + f(0,U(0)+V)V' = 0, \quad 0 \leq t \leq t_p \tag{9.13}$$

$$V(0) = \alpha - U(0) \ , \ V(t_p) = 0. \tag{9.14}$$

We solve this modified boundary layer correction problem (9.13) - (9.14) over the interval $0 \leq t \leq t_p$.

In order to solve the nonlinear two point boundary value problem given by the equations (9.13) - (9.14) [Boundary layer correction problem], we make use of the method of quasi-linearization (cf. Bellman and Kalaba [21], see also Lee [91], and Roberts and Shipman [131]) coupled with the classical second order central difference scheme (cf. Fox [57]). In fact any standard analytical or numerical method can be used.

After solving both the reduced and boundary layer correction problems, we combine the solutions of the reduced

problem (9.4) - (9.5) and the boundary layer correction problem (9.13) - (9.14) using the equation (9.12) to obtain an approximate solution to the original problem (9.1) - (9.2) over the interval $0 \leq x \leq 1$.

Repeat the process for various choices of t_p (terminal point of the boundary layer region), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|V(t)^{m+1} - V(t)^m| \leq \sigma \; ; \; 0 \leq t \leq t_p \qquad (9.15)$$

where

$V(t)^m$ = m^{th} iterate of the boundary layer region solution

and

σ = prescribed tolerance bound.

9.3 NUMERICAL EXAMPLES

To illustrate the boundary value method, we have applied it to three nonlinear singular perturbation problems. These examples have been chosen because they have been widely discussed in the literature and because approximate solutions are available for comparison.

Example 9.1 : Consider the following example from Bender and Orszag [22], Page : 463; Equations : 9.7.1 ;

$$\varepsilon y'' + 2y' + e^y = 0, \; 0 \leq x \leq 1 \qquad (9.16)$$

$$\text{with } y(0) = 0 \; , \; y(1) = 0. \qquad (9.17)$$

We have chosen to use Bender and Orszag's uniformly valid approximation (Bender and Orszag [22], Page : 463; Equation : 9.7.6) for comparison.

$$y(x) = \log \frac{2}{1+x} - (\exp(-2x/\varepsilon)) \log 2. \tag{9.18}$$

For this example, we have boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Bender and Orszag [22]).

The reduced problem is given by

$$2U' + e^{U} = 0, \tag{9.19}$$

$$U(1) = 0, \tag{9.20}$$

whose solution is

$$U(x) = \log \frac{2}{1+x} . \tag{9.21}$$

The boundary layer correction problem is given by

$$V'' + 2V' = 0 , \; 0 \leq t \leq t_p \tag{9.22}$$

$$V(0) = 0-U(0), \; V(t_p) = 0. \tag{9.23}$$

The computational results are presented in the Table 9.1, 9.2, for $\varepsilon = 10^{-2}$, 10^{-3}, respectively.

Example 9.2 : Now, consider the following example from Kevorkian and Cole [84], Page : 56; Equations : 2.5.1;

$$\varepsilon y'' + yy' - y = 0, \; 0 \leq x \leq 1 \tag{9.24}$$

$$\text{with } y(0) = -1 , \; y(1) = 3.9995. \tag{9.25}$$

We have chosen to use Kevorkian and Cole's uniformly valid approximation (Kevorkian and Cole [84], Pages : 57 and 58; Equations : 2.5.5, 2.5.11 and 2.5.14) for comparison.

$$y(x) = x + c_1 \tanh \frac{c_1}{2} (\frac{x}{\varepsilon} + c_2), \tag{9.26}$$

where

$$c_1 = 2.9995$$

and

$$c_2 = \frac{1}{c_1} \log (\frac{c_1-1}{c_1+1}).$$

For this example also we have a boundary layer of width $O(\varepsilon)$ at $x = 0$ (cf. Kevorkian and Cole [84], and Roberts [130]).

The reduced equation is given by

$$yy'-y = 0. \tag{9.27}$$

This implies two equations

$$y = 0, \tag{9.28}$$

$$y' = 1. \tag{9.29}$$

Only the equation (9.29) is the correct differential equation (cf. Kevorkian and Cole [84], and Roberts [130]). Hence the 'correct' reduced problem is given by

$$U' = 1, \tag{9.30}$$

$$U(1) = 3.9995, \tag{9.31}$$

whose solution is

$$U(x) = x+2.9995. \tag{9.32}$$

The boundary layer correction problem is given by

$$V'' + (U(0)+V)V' = 0, \; 0 \leq t \leq t_p \tag{9.33}$$

$$V(0) = -1-U(0), \; V(t_p) = 0. \tag{9.34}$$

The computational results are presented in the Table 9.3, 9.4, for $\varepsilon = 10^{-2}, 10^{-3}$, respectively.

Example 9.3 : Finally, consider the following example from O'Malley [113], Page : 9 ; Equations : 1.10; Case 2;

$$\varepsilon y'' - yy' = 0, \quad -1 \le x \le 1 , \tag{9.35}$$

$$\text{with } y(-1) = 0 , \quad y(1) = -1. \tag{9.36}$$

We have chosen to use O'Malley's approximate solution (O'Malley [113], Pages : 9 and 10; Equations : 1.13 and 1.14) for comparison.

$$y(x) = -\frac{[1-\exp(-(x+1)/\varepsilon)]}{[1+\exp(-(x+1)/\varepsilon)]} . \tag{9.37}$$

For this example, we have a boundary layer of width $O(\varepsilon)$ at the left end of the interval, that is, at $x = -1$ (cf. O'Malley [113], and Roberts [129]).

The reduced equation is given by

$$-yy' = 0. \tag{9.38}$$

This implies two equations

$$y = 0, \tag{9.39}$$

$$y' = 0. \tag{9.40}$$

Only the equation (9.40) is the correct differential equation (cf. O'Malley [113], and Roberts [129]). Hence the 'correct' reduced problem is given by

$$U' = 0 , \tag{9.41}$$

$$U(1) = -1, \tag{9.42}$$

whose solution is

$$U(x) = -1. \tag{9.43}$$

We remark that in this example the stretching transformation is given by

$$x = t\varepsilon - 1. \tag{9.44}$$

The boundary layer correction problem is given by

$$V'' - (U(-1)+V)V' = 0, \; 0 \leq t \leq t_p \tag{9.45}$$

$$V(0) = 0-U(-1), \; V(t_p) = 0. \tag{9.46}$$

The computational results are presented in the Table 9.5, 9.6, for $\varepsilon = 10^{-2}$, 10^{-3}, respectively.

9.4 DISCUSSION

As mentioned, the method is iterative on the terminal point of the boundary layer region. The process is to be repeated for various choices of t_p (terminal point of the boundary layer region), until the solution profiles stabilize. The point t_p is not unique but can assume a wide range of values. To reduce the amount of computation we desire the smallest value of t_p which gives the required accuracy. Because the boundary layer region is small relative to the entire interval of the original problem, we can usually improve our accuracy by making t_p larger. To solve the nonlinear two point boundary value problems in our numerical experimentation, we have used the method of quasilinearization (cf. Bellman and Kalaba [21] coupled with the classical second order central difference scheme

(cf. Fox [57]). In fact any standard analytical or numerical method can be used.

The boundary value method is similar in some respects to the asymptotic expansion methods but differs in detail. In fact the methods differ in how they use the data which are available. In our prescription for the boundary value method we have assumed that the solution of reduced problem can be found. While our orientation is toward numerical methods using a computer, there is no reason that other solution techniques cannot be used : analytical approximation, or even asymptotic expansion methods.

We have implemented the present method on three examples by taking different values for ε. We have tabulated the computational results obtained by the boundary value method as well as the approximate solutions developed by others. It can be observed from the tables that present method approximates the exact solution very well.

9.5 SOME THEORETICAL RESULTS

If we look at the boundary value method for linear problems, we find some interesting results. To see these, let us consider the following linear singular perturbation problem :

$$\varepsilon y'' + f(x)y' + g(x)y = 0, \quad 0 \leq x \leq 1 \tag{9.47}$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta, \tag{9.48}$$

where ε is a small positive parameter ($0 < \varepsilon << 1$); α, β are given constants; $f(x)$ and $g(x)$ are assumed to be sufficiently continuously differentiable functions on $[0,1]$. Furthermore, we assume that $f(x) \geq M > 0$ throughout the interval $[0,1]$. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$. The solution of problem (9.47) - (9.48) admits the representation

$$y = U(x)+V(t) + O(\varepsilon) \text{ as } \varepsilon \to 0^+ \tag{9.49}$$

uniformly in $[0,1]$, with U and V are respectively the solutions of reduced problem and the boundary layer correction problem. That is

$$f(x)U' + g(x)U = 0 \tag{9.50}$$

$$\text{with } U(1) = \beta \tag{9.51}$$

and

$$V'' + f(0)V' = 0 \text{ , } t > 0 \tag{9.52}$$

$$\text{with } V(0) = \alpha-U(0), \quad \lim_{t\to+\infty} V(t) = 0. \tag{9.53}$$

Since from the theory of singular perturbations, we know that V, the solution of boundary layer correction problem (9.52) - (9.53) is insignificant outside the boundary layer region, we simply consider the following modified boundary layer correction problem :

$$V'' + f(0)V' = 0, \; 0 \leq t \leq t_p \tag{9.54}$$

$$\text{with } V(0) = \alpha-U(0), \; V(t_p) = 0 \tag{9.55}$$

where t_p is a terminal point or width or thickness of the boundary layer. Again, from the theory of singular perturbations we know that U, the solution of the reduced problem (9.50) - (9.51) is constant inside the boundary layer region. That is in the boundary layer region

$$U(x) \sim U(0) \quad \text{as } \varepsilon \to 0^+ \tag{9.56a}$$

$$\text{i.e.,} \quad \beta \exp\left[\int_x^1 \frac{g(s)}{f(s)}\, ds\right] \sim \beta \exp\left[\int_0^1 \frac{g(s)}{f(s)}\, ds\right] \tag{9.56b}$$

for $0 \leq x \leq x_p$ as $\varepsilon \to 0^+$. Hence, in place of the boundary layer correction problem (9.54) - (9.55), we get simply the following boundary layer problem (by making use of the transformation $x = t\varepsilon$ and then rescaling

$$y(x) = y(t\varepsilon) = Y(t) \tag{9.57a}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{9.57b}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{9.57c}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{9.57d}$$

as in Chapter 5) :

$$Y''(t) + F(0)Y'(t) = 0, \quad 0 \leq t \leq t_p \tag{9.58}$$

$$\text{with } Y(0) = \alpha, \; Y(t_p) = U(x_p) = \bar{\alpha} \text{ (say).} \tag{9.59}$$

Observe that this is identically the inner region problem of the Chapter 5 (compare with equations (5.12) - (5.11)).

Thus in a manner of speaking, we have shown that the boundary layer correction problem of this Chapter 9 and inner region problem of Chapter 5 are equivalent, as long as U, the solution of the reduced problem can be considered to be constant in the boundary layer region $[0, x_p]$.

To gain further insight into the choice of the t_p, terminal point of the boundary layer region, let us carry our discussion on linear problem (9.47) - (9.48) a little further. As mentioned several times, we choose the t_p as

$$t_p << \frac{1}{\varepsilon} . \tag{9.60}$$

Let us now derive lower bound for the t_p. To get this we impose the condition that the t_p is to be chosen such that the difference between the solution (say $\bar{V}$) of problem (9.54) - (9.55) and the solution (say V) of problem (9.52) - (9.53) is $O(\varepsilon)$ or tends to zero as $\varepsilon \to 0^+$. Expressed mathematically,

$$|\bar{V}-V| = O(\varepsilon) \text{ for } 0 \leq t \leq t_p \tag{9.61}$$

For problems (9.52) - (9.53) and (9.54) - (9.55) we can explicitly find the solutions which are respectively

$$V(t) = (\alpha-U(0))+(\alpha-U(0))(e^{-f(0)t}-1) = (\alpha-U(0))\,e^{-f(0)t} \tag{9.62}$$

and

$$\bar{V}(t) = (\alpha-U(0)) + \frac{(\alpha-U(0))}{(e^{f(0)t_p} - 1)}\, e^{f(0)t_p}(e^{-f(0)t} - 1) \tag{9.63}$$

Thus

$$\bar{V}-V = \frac{(\alpha-U(0))}{(e^{f(0)t_p} - 1)} (e^{-f(0)t} - 1). \tag{9.64}$$

Note that $|e^{-f(0)t} - 1| \leq 1$ for all $t \geq 0$ so we obtain

$$|\bar{V}-V| \leq \frac{|\alpha-U(0)|}{(e^{f(0)t_p} - 1)}. \tag{9.65}$$

In order to have

$$\frac{|\alpha-U(0)|}{(e^{f(0)t_p} - 1)} \leq |\alpha-U(0)|\varepsilon \tag{9.66}$$

we choose t_p such that

$$t_p \geq \frac{1}{f(0)} (\log 2 - \log \varepsilon). \tag{9.67}$$

In particular, if $\varepsilon = 10^{-\gamma}$, $\gamma > 0$, the lower bound for t_p takes the form

$$t_p \geq \frac{1}{f(0)} (0.69 + 2.3\gamma). \tag{9.68}$$

In general, including some nonlinear problems, (as suggested by Hsiao and Jordan [74], and Lorenz [95]), one may determine t_p without computing the explicit solutions of problems (9.52) - (9.53) and (9.54) - (9.55). This is done by considering an inequality such as

$$e^{-f(0)t_p} \leq \varepsilon. \tag{9.69}$$

For $\varepsilon = 10^{-\gamma}$, one obtains from (9.69) the crude estimate

$$t_p \geq \frac{\gamma \log 10}{f(0)} \cong \frac{3\gamma}{f(0)}. \tag{9.70}$$

Thus we can summarize the above results as follows.

Result : Let $V, \bar{V}$ be the solutions of problems (9.52) - (9.53) and (9.54) - (9.55) respectively. Let t_p be chosen such that it satisfies the above conditions then

$$|V-\bar{V}| = O(\varepsilon) \quad \text{for } 0 \leq t \leq t_p$$

i.e. $|V-\bar{V}| \to 0$ as $\varepsilon \to 0^+$ for $0 \leq t \leq t_p$.

9.6 CONCLUSIONS

We have described the boundary value method for solving a class of nonlinear singular perturbation problems. It provides an alternative and supplementary method to the conventional ways of solving certain classes of nonlinear singular perturbation problems. It is a practical method, easily implemented on a computer to solve certain classes of nonlinear singular perturbation problems with a modest amount of problem preparation. We have illustrated the method with three examples with known solutions and have demonstrated that the boundary value method approximates the exact solution well.

Table 9.1

Computational results for Example 9.1, $\varepsilon = 10^{-2}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Bender and Orszag's solution [22]
0.0	0.0000000	0.0000000	0.0000000
$5.0(10^{-3})$	0.4340381	0.4340188	0.4331651
$1.0(10^{-2})$	0.5900430	0.5900167	0.5893896
$2.5(10^{-2})$	0.6638921	0.6638619	0.6637842
$5.0(10^{-2})$	<u>0.6443570</u>	0.6443266	0.6443255
$7.5(10^{-2})$		0.6208263	0.6208263
$1.0(10^{-1})$		<u>0.5978370</u>	0.5978370
0.2			0.5108256
0.3	0.4307829	0.4307829	0.4307829
0.4	0.3566749	0.3566749	0.3566749
0.5	0.2876821	0.2876821	0.2876821
0.6	0.2231435	0.2231435	0.2231435
0.7	0.1625189	0.1625189	0.1625189
0.8	0.1053605	0.1053605	0.1053605
0.9	0.0512933	0.0512933	0.0512933
1.0	0.0000000	0.0000000	0.0000000

Table 9.2

Computational results for Example 9.1, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Bender and Orszag's solution [22]
0.0	0.0000000	0.0000000	0.0000000
$5.0(10^{-4})$	0.4385258	0.4385065	0.4376527
$1.0(10^{-3})$	0.5989939	0.5989676	0.5983404
$2.5(10^{-3})$	0.6860878	0.6860576	0.6859799
$5.0(10^{-3})$	0.6881596	0.6881292	0.6881282
$7.5(10^{-3})$		0.6856750	0.6856750
$1.0(10^{-2})$		0.6831968	0.6831968
0.2			0.5108256
0.3	0.4307829	0.4307829	0.4307829
0.4	0.3566749	0.3566749	0.3566749
0.5	0.2876821	0.2876821	0.2876821
0.6	0.2231435	0.2231435	0.2231435
0.7	0.1625189	0.1625189	0.1625189
0.8	0.1053605	0.1053605	0.1053605
0.9	0.0512933	0.0512933	0.0512933
1.0	0.0000000	0.0000000	0.0000000

Table 9.3

Computational results for Example 9.2, $\varepsilon = 10^{-2}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Kevorkian and Cole's solution [84]
0.0	-1.0000000	-1.0000000	-1.0000000
$5.0(10^{-3})$	1.1675228	1.1675186	1.1529593
$1.0(10^{-2})$	2.4771085	2.4771040	2.4659397
$2.5(10^{-2})$	3.0182330	3.0182297	3.0178623
$5.0(10^{-2})$	<u>3.0495000</u>	3.0494967	3.0494963
$7.5(10^{-2})$		3.0745000	3.0745000
$1.0(10^{-1})$		<u>3.0995000</u>	3.0995000
0.2			3.1995000
0.3	3.2995000	3.2995000	3.2995000
0.4	3.3995000	3.3995000	3.3995000
0.5	3.4995000	3.4995000	3.4995000
0.6	3.5995000	3.5995000	3.5995000
0.7	3.6995000	3.6995000	3.6995000
0.8	3.7995000	3.7995000	3.7995000
0.9	3.8995000	3.8995000	3.8995000
1.0	3.9995000	3.9995000	3.9995000

Table 9.4

Computational results for Example 9.2, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	5 $y(x)$	10 $y(x)$	Kevorkian and Cole's solution [84]
0.0	-1.0000000	-1.0000000	-1.0000000
$5.0(10^{-4})$	1.1630228	1.1630186	1.1484593
$1.0(10^{-3})$	2.4681085	2.4681040	2.4569397
$2.5(10^{-3})$	2.9957330	2.9957297	2.9953623
$5.0(10^{-3})$	<u>3.0045000</u>	3.0044967	3.0044963
$7.5(10^{-3})$		3.0070000	3.0070000
$1.0(10^{-2})$		<u>3.0095000</u>	3.0095000
0.2			3.1995000
0.3	3.2995000	3.2995000	3.2995000
0.4	3.3995000	3.3995000	3.3995000
0.5	3.4995000	3.4995000	3.4995000
0.6	3.5995000	3.5995000	3.5995000
0.7	3.6995000	3.6995000	3.6995000
0.8	3.7995000	3.7995000	3.7995000
0.9	3.8995000	3.8995000	3.8995000
1.0	3.9995000	3.9995000	3.9995000

Table 9.5

Computational results for Example 9.3, $\varepsilon = 10^{-2}$

t_p → x	5 y(x)	10 y(x)	O'Malley's solution [113]
-1.000	0.0000000	0.0000000	0.0000000
-0.995	-0.2512424	-0.2451582	-0.2449188
-0.990	-0.4733490	-0.4625226	-0.4621171
-0.975	-0.8637724	-0.8486840	-0.8482836
-0.950	-1.0000000	-0.9867644	-0.9866143
-0.925		-0.9989920	-0.9988944
-0.900		-1.0000000	-0.9999092
-0.8			-1.0000000
-0.6	-1.0000000	-1.0000000	-1.0000000
-0.4	-1.0000000	-1.0000000	-1.0000000
-0.2	-1.0000000	-1.0000000	-1.0000000
0.0	-1.0000000	-1.0000000	-1.0000000
0.2	-1.0000000	-1.0000000	-1.0000000
0.4	-1.0000000	-1.0000000	-1.0000000
0.6	-1.0000000	-1.0000000	-1.0000000
0.8	-1.0000000	-1.0000000	-1.0000000
1.0	-1.0000000	-1.0000000	-1.0000000

Table 9.6

Computational results for Example 9.3, $\varepsilon = 10^{-3}$

t_p → x	5 y(x)	10 y(x)	O'Malley's solution [113]
-1.0000	0.0000000	0.0000000	0.0000000
-0.9995	-0.2512424	-0.2451582	-0.2449191
-0.9990	-0.4733490	-0.4625226	-0.4621180
-0.9975	-0.8637724	-0.8486840	-0.8482833
-0.9950	<u>-1.0000000</u>	-0.9867644	-0.9866143
-0.9925		-0.9989920	-0.9988944
-0.9900		<u>-1.0000000</u>	-0.9999092
-0.8			-1.0000000
-0.6	-1.0000000	-1.0000000	-1.0000000
-0.4	-1.0000000	-1.0000000	-1.0000000
-0.2	-1.0000000	-1.0000000	-1.0000000
0.0	-1.0000000	-1.0000000	-1.0000000
0.2	-1.0000000	-1.0000000	-1.0000000
0.4	-1.0000000	-1.0000000	-1.0000000
0.6	-1.0000000	-1.0000000	-1.0000000
0.8	-1.0000000	-1.0000000	-1.0000000
1.0	-1.0000000	-1.0000000	-1.0000000

CHAPTER 10

A NONASYMPTOTIC METHOD FOR GENERAL LINEAR SINGULAR PERTURBATION PROBLEMS

10.1 INTRODUCTION

In Chapter 3, we developed a nonasymptotic method for solving linear singularly perturbed two-point boundary-value problems with a left-end boundary layer. The motivating impulse for this method was to provide the practicing engineer or applied mathematician a means of solving a class of singular perturbation problems in a routine manner. The method avoided the principal problem of the conventional techniques, namely, finding the appropriate asymptotic expansion. As part of a continuing effort to determine the applicability and the limitations of the nonasymptotic method, we have been attempting to solve general linear singularly perturbed two-point boundary-value problems in ordinary differential equations. Singular perturbation problems with a right-end boundary layer, singular perturbation problems with an interior layer, and singular perturbation problems with two boundary layers, are worthy contenders. For a detailed discussion on the analytic theory of general singular perturbation problems, one may refer to O'Malley [113], and Kevorkian and Cole [84].

To make the chapter self-contained, we briefly review the nonasymptotic method for solving singular perturbation problems with a left-end boundary layer. Than, we extend the method to solve general linear singularly perturbed two-point boundary-value problems. Firstly, we discuss problems with a right-end boundary layer. Secondly, we discuss problems with an interior layer. Finally, we discuss problems with two boundary layers. To support our analytical development, we present the numerical results of some model problems.

10.2 NONASYMPTOTIC METHOD, LEFT-END BOUNDARY LAYER

To recapitulate the nonasymptotic method, we consider the following singular perturbation problem :

$$\varepsilon y''(x) + a(x)y'(x)+b(x)y(x) = f(x), \; 0 \leq x \leq 1 \qquad (10.1)$$

$$\text{with } y(0) = \alpha \text{ and } y(1) = \beta \qquad (10.2)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$; α, β are given constants; $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

Let δ be a small positive deviating argument $(0 < \delta << 1)$. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x-\delta) \approx y(x)-\delta y'(x) + \frac{\delta^2}{2} y''(x) \qquad (10.3)$$

and consequently, the equation (10.1) is replaced by the following first-order differential equation with a small deviating argument :

$$y'(x) = p(x)y(x-\delta) + q(x)y(x) + r(x) \tag{10.4}$$

for $\delta \leq x \leq 1$ where

$$p(x) = \frac{-2\varepsilon}{2\varepsilon\delta + \delta^2 a(x)} \tag{10.5}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{2\varepsilon\delta + \delta^2 a(x)} \tag{10.6}$$

$$r(x) = \frac{\delta^2 f(x)}{2\varepsilon\delta + \delta^2 a(x)} \,. \tag{10.7}$$

Transition from the equation (10.1) to equation (10.4) is admitted, because of the condition that δ is small ($0 < \delta << 1$). We solve the equation (10.4) by employing the Simpson's rule coupled with the discrete invariant imbedding algorithm. We repeat the process for different choices of the deviating argument, until the solution profiles stabilizes. Further details can be found in Chapter 3.

10.3 NONASYMPTOTIC METHOD, RIGHT-END BOUNDARY LAYER

We now assume that $a(x) \leq M < 0$ throughout the interval $[0,1]$, where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 1$.

The evaluation of the right-end boundary layer for (10.1)-(10.2) is similar to that of the left-end boundary layer,

but there are some differences worth nothing. By using Taylor series expansion in the neighbourhood of the point x, we have

$$y(x+\delta) \approx y(x) + \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{10.8}$$

and consequently, the equation (10.1) is replaced by the following first-order differential equation with a small deviating argument :

$$2\varepsilon y(x+\delta) - 2\varepsilon y(x) - 2\varepsilon\delta y'(x) + \delta^2 a(x)y'(x)$$
$$+ \delta^2 b(x)y(x) = \delta^2 f(x) \tag{10.9}$$

Transition from the equation (10.1) to equation (10.9) is again admitted, because of the condition that δ is small ($0 < \delta << 1$). We rewrite the equation (10.9) in the following convenient form :

$$y'(x) = p(x)y(x+\delta) + q(x)y(x) + r(x) \tag{10.10}$$

for $0 \leq x \leq 1-\delta$ where

$$p(x) = \frac{-2\varepsilon}{\delta^2 a(x) - 2\varepsilon\delta} \tag{10.11}$$

$$q(x) = \frac{2\varepsilon - \delta^2 b(x)}{\delta^2 a(x) - 2\varepsilon\delta} \tag{10.12}$$

$$r(x) = \frac{\delta^2 f(x)}{\delta^2 a(x) - 2\varepsilon\delta} . \tag{10.13}$$

We will now describe the numerical scheme for solving the equation (10.10). As usual, we divide the interval $[0,1]$ into N equal parts with mesh h,

i.e., $h = \frac{1}{N}$ and $x_i = ih$ for $i = 0, 1, \ldots, N$.

Integrating the equation (10.10) in $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, N-1$; we get

$$y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} [p(x)y(x+\delta) + q(x)y(x) + r(x)]\, dx. \tag{10.14}$$

By making use of the Simpson's rule for evaluating the integrals approximately, we obtain

$$\begin{aligned} y(x_i) - y(x_{i-1}) = {} & \frac{h}{6}[p(x_{i-1})y(x_{i-1}+\delta)] \\ & + \frac{h}{6}[4p(x_{i-1/2})y(x_{i-1/2}+\delta)] \\ & + \frac{h}{6}[p(x_i)y(x_i+\delta)] \\ & + \frac{h}{6}[q(x_{i-1})y(x_{i-1})] \\ & + \frac{h}{6}[4q(x_{i-1/2})y(x_{i-1/2})] \\ & + \frac{h}{6}[q(x_i)y(x_i)] \\ & + \frac{h}{6}[r(x_{i-1}) + 4r(x_{i-1/2}) + r(x_i)]. \end{aligned} \tag{10.15}$$

By means of Taylor series expansion, we have

$$y(x+\delta) \approx y(x) + \delta y'(x)$$

and, then by approximating $y'(x)$ by interpolation formula, we get

$$y(x_i+\delta) \approx (1 - \frac{\delta}{h})y(x_i) + \frac{\delta}{h}\, y(x_{i+1}), \tag{10.16}$$

$$y(x_{i-1}+\delta) \approx (1 - \frac{\delta}{h})y(x_{i-1}) + \frac{\delta}{h}\, y(x_i), \tag{10.17}$$

and

$$y(x_{i-1/2}+\delta) \approx y(x_{i-1/2}) + \frac{\delta}{h}\, y(x_i) - \frac{\delta}{h}\, y(x_{i-1}). \tag{10.18}$$

Hence, by making use of equations (10.16), (10.17), and (10.18) in the equation (10.15) leads to

$$y(x_i)-y(x_{i-1}) = \left[\frac{h-\delta}{6}\, p(x_{i-1}) - \frac{4\delta}{6}\, p(x_{i-1/2}) + \frac{h}{6}\, q(x_{i-1})\right] y(x_{i-1})$$

$$+\left[\frac{\delta}{6}\, p(x_{i-1}) + \frac{4\delta}{6}\, p(x_{i-1/2}) + \frac{h-\delta}{6}\, p(x_i) + \frac{h}{6}\, q(x_i)\right] y(x_i)$$

$$+\left[\frac{\delta}{6}\, p(x_i)\right] y(x_{i+1})$$

$$+\left[\frac{4h}{6}\, p(x_{i-1/2}) + \frac{4h}{6}\, q(x_{i-1/2})\right] y(x_{i-1/2})$$

$$+\frac{h}{6}\left[r(x_{i-1}) + 4r(x_{i-1/2}) + r(x_i)\right]. \qquad (10.19)$$

Again, by using Taylor's theorem it is easy to show that

$$y(x_{i-1/2}) = \frac{y(x_{i-1})+y(x_i)}{2} + \frac{h}{8}\left[y'(x_{i-1})-y'(x_i)\right] + O(h^4). \qquad (10.20)$$

In view of the equation (10.10) and the above (10.20) we get

$$y(x_{i-1/2}) \approx \frac{1}{2}\, y(x_{i-1}) + \frac{1}{2}\, y(x_i)$$

$$+\frac{h}{8}\left[p(x_{i-1})y(x_{i-1}+\delta)+q(x_{i-1})y(x_{i-1})+r(x_{i-1})\right]$$

$$-\frac{h}{8}\left[p(x_i)y(x_i+\delta)+q(x_i)y(x_i)+r(x_i)\right]. \qquad (10.21)$$

By making use of (10.21) in (10.19) we get

$$
\begin{aligned}
y(x_i)-y(x_{i-1}) = {} & \Big[\frac{h-\delta}{6}\, p(x_{i-1}) - \frac{4\delta}{6}\, p(x_{i-1/2}) + \frac{h}{6}\, q(x_{i-1}) \\
& + \left(\frac{1}{2}+\frac{h}{8}q(x_{i-1})\right)\left(\frac{4h}{6}\, p(x_{i-1/2}) + \frac{4h}{6}\, q(x_{i-1/2})\right)\Big] y(x_{i-1}) \\
& + \Big[\frac{\delta}{6}p(x_{i-1}) + \frac{4\delta}{6}\, p(x_{i-1/2}) + \frac{h-\delta}{6}p(x_i) + \frac{h}{6}\, q(x_i) \\
& + \left(\frac{1}{2}-\frac{h}{8}q(x_i)\right)\left(\frac{4h}{6}p(x_{i-1/2}) + \frac{4h}{6}q(x_{i-1/2})\right)\Big]\ y(x_i) \\
& + \Big[\frac{\delta}{6}\, p(x_i)\Big]\ y(x_{i+1}) \\
& + \Big[\frac{h}{8}p(x_{i-1})\left(\frac{4h}{6}p(x_{i-1/2}) + \frac{4h}{6}q(x_{i-1/2})\right)\Big]\ y(x_{i-1}+\delta) \\
& - \Big[\frac{h}{8}p(x_i)\left(\frac{4h}{6}p(x_{i-1/2}) + \frac{4h}{6}\, q(x_{i-1/2})\right)\Big]\ y(x_i+\delta) \\
& + \Big[\frac{1}{6} + \frac{1}{8}\left(\frac{4h}{6}\, p(x_{i-1/2}) + \frac{4h}{6}\, q(x_{i-1/2})\right)\Big]\ hr(x_{i-1}) \\
& + \Big[\frac{1}{6} - \frac{1}{8}\left(\frac{4h}{6}\, p(x_{i-1/2}) + \frac{4h}{6}\, q(x_{i-1/2})\right)\Big]\ hr(x_i) \\
& + \frac{4h}{6}\, r(x_{i-1/2}).
\end{aligned} \tag{10.22}
$$

Finally, by making use of equations (10.16) and (10.17) in the equation (10.22) leads after simple manipulation (rearrangement) to the following three-term recurrence relationship :

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \tag{10.23}$$

for $i = 1, 2, \ldots, N-1$; where

$$E_i = [1 + \frac{h-\delta}{6} p_{i-1} - \frac{4\delta}{6} p_{i-1/2} + \frac{h}{6} q_{i-1}$$

$$+ (\frac{1}{2} + \frac{h-\delta}{8} p_{i-1} + \frac{h}{8} q_{i-1})(\frac{4h}{6} p_{i-1/2} + \frac{4h}{6} q_{i-1/2})] \quad (10.24)$$

$$F_i = [1 - \frac{\delta}{6} p_{i-1} - \frac{4\delta}{6} p_{i-1/2} - \frac{h-\delta}{6} p_i - \frac{h}{6} q_i$$

$$+(- \frac{1}{2} - \frac{\delta}{8} p_{i-1} + \frac{h-\delta}{8} p_i + \frac{h}{8} q_i)(\frac{4h}{6} p_{i-1/2} + \frac{4h}{6} q_{i-1/2})]$$

$$(10.25)$$

$$G_i = [(\frac{1}{6} - \frac{1}{8} (\frac{4h}{6} p_{i-1/2} + \frac{4h}{6} q_{i-1/2}))\ \delta p_i] \quad (10.26)$$

$$H_i = -[\frac{1}{6} + \frac{1}{8} (\frac{4h}{6} p_{i-1/2} + \frac{4h}{6} q_{i-1/2})]\ hr_{i-1}$$

$$-[\frac{1}{6} - \frac{1}{8} (\frac{4h}{6} p_{i-1/2} + \frac{4h}{6} q_{i-1/2})]\ hr_i$$

$$- \frac{4h}{6} r_{i-1/2} \quad (10.27)$$

and $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$, and $r_i = r(x_i)$. Equation (10.23) gives a system of (N-1) equations with (N+1) unknowns y_0 to y_N. The two given boundary conditions (10.2) together with these (N-1) equations are then sufficient to solve for the unknowns y_i's. The matrix problem associated with the equation (10.23) is tridiagonal algebraic system and the solution of this tridiagonal system can easily be obtained by using an efficient algorithm called discrete invariant imbedding (Angel and Bellman [4]).

Repeat the process for different choices of δ (deviating argument, satisfying the condition, $0 < \delta << 1$), until the

of the nonasymptotic method for solving singular perturbation problems with an interior layer.

Problem 10.2 : Consider the following singular perturbation problem :

$$\varepsilon y''(x) + xy'(x) - y(x) = 0, \; -1 \leq x \leq 1 \qquad (10.31)$$

$$\text{with } y(-1) = 1 \text{ , and } y(1) = 2. \qquad (10.32)$$

Equation (10.31) is in the form of (10.1) with $a(x) = x$, $b(x) = -1$, and $f(x) = 0$. For this problem, we have an interior layer of width $O(\sqrt{\varepsilon})$ at $x = 0$ (For details, see O'Malley [113], Pages : 168-172; Equations : 8.1; Case : i, and Kevorkian and Cole [84], Pages : 41-43; Equations : 2.3.76 and 2.3.77).

We see that the function

$$a(x) = x < 0 \text{ for } -1 \leq x < 0, \qquad (10.33)$$

$$a(x) = x = 0 \text{ for } x = 0, \qquad (10.34)$$

$$a(x) = x > 0 \text{ for } 0 < x \leq 1. \qquad (10.35)$$

Hence, by making use of transitions (Equations(10.3) and (10.8)) suggested for left-end and right-end boundary layers, we replace the equation (10.31) by the following first-order differential equations with a small deviating argument :

$$y'(x) = p(x)y(x+\delta)+q(x)y(x)+r(x) \text{ for } -1 \leq x \leq -\delta \qquad (10.36)$$

where $p(x)$, $q(x)$, and $r(x)$ are given by (10.11) - (10.13),

$$y'(x) = p(x)y(x-\delta)+q(x)y(x)+r(x) \text{ for } \delta \leq x \leq 1, \qquad (10.37)$$

where $p(x)$, $q(x)$ and $r(x)$ are given by (10.5) - (10.7).

We now divide the interval $[-1, 1]$ into N equal parts with mesh size h,

i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0, 1, \ldots, N$.

Let us denote $\frac{N}{2} = L$. Then, integrating

the equation (10.36) in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, L-1$;

and

the equation (10.37) in $[x_i, x_{i+1}]$ for $i = L+1, L+2, \ldots, N-1$;

we get a system of (N-2) equations with (N+1) unknowns.

From the given boundary conditions (10.32) we get two equations

$$y_0 = y(-1) = 1 , \tag{10.38}$$

$$y_N = y(1) = 2. \tag{10.39}$$

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots, y_N)$. For this, we consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$, we get the following:

$$\varepsilon y''(x_L) + b(x_L)y(x_L) = f(x_L). \tag{10.40}$$

By making use of the second-order central finite difference formula, we get the following equation from (10.40) :

$$[\varepsilon]\, y_{L-1} - [2\varepsilon - h^2 b_L]\, y_L + [\varepsilon]\, y_{L+1} = h^2 f_L. \tag{10.41}$$

With this equation (10.41), we now have (N+1) equations to solve for the unknowns $(y_0, y_1, \ldots, y_N)$.

The process is to be repeated for various choices of δ, until the solution profiles do not differ materially from

iteration to iteration. The numerical results for Problem 10.2 are presented in the Table 10.3, 10.4 for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

10.5 NONASYMPTOTIC METHOD, TWO BOUNDARY LAYERS

The suggestions given for interior layer problems apply mutatis mutandis to problems with two boundary layers. To illustrate this, we will again consider the case where $a(x)$ changes sign in the domain of interest. Without loss of generality, we take $a(0) = 0$, and the interval to be $[-1,1]$. Again, with the help of one model problem we demonstrate the applicability of the nonasymptotic method for solving singular perturbation problems with two boundary layers.

Problem 10.3 : Consider the following singular perturbation problem :

$$\varepsilon y''(x) - xy'(x) - y(x) = 0; \; -1 \leq x \leq 1 \qquad (10.42)$$

$$\text{with } y(-1) = 1, \text{ and } y(1) = 2. \qquad (10.43)$$

Equation (10.42) is in the form of (10.1) with $a(x) = -x$, $b(x) = -1$ and $f(x) = 0$. For this problem, we have two boundary layers, one at $x = -1$, and one at $x = 1$ (For details, see O'Malley [113], Pages : 168-173; Equations : 8.1; Case : ii). We see that the function

$$a(x) = -x > 0 \quad \text{for } -1 \leq x < 0, \qquad (10.44)$$

$$a(x) = -x = 0 \quad \text{for } \; x = 0, \qquad (10.45)$$

$$a(x) = -x < 0 \quad \text{for } \; 0 < x \leq 1. \qquad (10.46)$$

Hence, by making use of transitions (Equations (10.3) and (10.8)) suggested for left-end and right-end boundary layers, we replace the equation (10.42), by the following first-order differential equations with a small deviating argument :

$$y'(x) = p(x)y(x-\delta)+q(x)y(x)+r(x) \quad \text{for } -1+\delta \leq x \leq 0, \tag{10.47}$$

where $p(x)$, $q(x)$, and $r(x)$ are given by (10.5)-(10.7),

$$y'(x) = p(x)y(x+\delta)+q(x)y(x)+r(x) \quad \text{for } 0 \leq x \leq 1-\delta , \tag{10.48}$$

where $p(x)$, $q(x)$, and $r(x)$ are given by (10.11)-(10.13).

As usual, we divide the interval $[-1,1]$ into N equal parts with mesh size h,

i.e., $h = \frac{2}{N}$ and $x_i = -1 + ih$ for $i = 0,1,\dots,N$.

Let us denote $\frac{N}{2} = L$. Then, integrating

the equation (10.47) in $[x_i, x_{i+1}]$ for $i = 1, 2,\dots,L-1$;

and

the equation (10.48) in $[x_{i-1}, x_i]$ for $i = L+1, L+2,\dots,N-1$;

we get a system of (N-2) equations with (N+1) unknowns. From the boundary conditions (10.43) we have two equations

$$y_0 = y(-1) = 1 , \tag{10.49}$$

$$y_N = y(1) = 2. \tag{10.50}$$

We need one more equation to solve for the unknowns. As in the previous case, we again consider the original equation at $x = x_L = 0$. Since $a(x) = 0$ at $x = x_L = 0$, it leads to

$$\varepsilon y''(x_L) + b(x_L)\, y(x_L) = f(x_L). \tag{10.51}$$

Again by using the second-order central finite difference formula, we get the following equation from (10.51) :

$$[\varepsilon]\; y_{L-1} - [2\varepsilon - h^2 b_L]\; y_L + [\varepsilon]\; y_{L+1} = h^2 f_L. \qquad (10.52)$$

Hence, with this equation (10.52), we now have (N+1) equations to solve for the unknowns $(y_0, y_1, \ldots, y_N)$.

The process is to be repeated for various choices of δ, until the solution profiles stabilize. The numerical results for Problem 10.3 are presented in the Table 10.5, 10.6, for $\varepsilon = 10^{-3}, 10^{-4}$, respectively.

10.6 DISCUSSION

As mentioned, the method is iterative on the deviating argument δ. The process is to be repeated for different choices of δ (deviating argument), until the solution profiles do not differ materially from iteration to iteration. The choice of δ is not unique but can assume any number of values satisfying the condition, $0 < \delta << 1$. To reduce the amount of computation, we fix the mesh size h and vary the deviating argument δ. Finally, we pick up the smallest value of δ which produces the required accuracy. We have implemented this method on three model problems, a singular perturbation problem with a right-end boundary layer, a singular perturbation problem with an interior layer, and a singular perturbation problem with two boundary layers, by taking different values for ε. The numerical results are presented in Tables 10.1 - 10.6. We have

given here only a few values although the solutions are computed at all the points with mesh size h. It can be observed from the tables that the present method approximates the exact solution very well. This shows the efficiency and accuracy of the present method.

10.7 CONCLUSIONS

We have shown that the nonasymptotic method is capable of solving general linear singularly perturbed two-point boundary value problems. This method provides an alternative and supplementary technique to the conventional ways of solving singular perturbation problems. It is a practical method, easily implemented on a computer to solve singular perturbation problems with a modest amount of problem preparation. We have illustrated the method with three model problems and have demonstrated that the nonasymptotic method approximates the exact solution well.

Table 10.1

Numerical results for Problem 10.1, $\varepsilon = 10^{-3}$ and $h = 0.01$

δ → x	ε y(x)	5ε y(x)	10ε y(x)
0.00	1.0000000	1.0000000	1.0000000
0.10	0.9999999	0.9999999	1.0000000
0.20	0.9999999	0.9999997	1.0000000
0.30	0.9999999	0.9999996	1.0000000
0.40	0.9999999	0.9999995	1.0000000
0.50	0.9999999	0.9999994	1.0000000
0.60	0.9999999	0.9999993	1.0000000
0.70	0.9999999	0.9999992	1.0000000
0.80	0.9999999	0.9999991	1.0000000
0.90	0.9999990	0.9999990	1.0000000
0.91	0.9999961	0.9999990	1.0000000
0.92	0.9999847	0.9999990	1.0000000
0.93	0.9999389	0.9999990	1.0000000
0.94	0.9997558	0.9999989	1.0000000
0.95	0.9990234	0.9999980	0.9999999
0.96	0.9960937	0.9999837	0.9999989
0.97	0.9843749	0.9997548	0.9999664
0.98	0.9375000	0.9960927	0.9989594
0.99	0.7500000	0.9374990	0.9677419
1.00	0.0000000	0.0000000	0.0000000

Table 10.2

Numerical results for Problem 10.1, $\varepsilon = 10^{-4}$ and $h = 0.01$

$\delta \rightarrow$ x	ε y(x)	5ε y(x)	10ε y(x)
0.00	1.0000000	1.0000000	1.0000000
0.10	1.0000054	1.0000001	1.0000001
0.20	1.0000108	1.0000002	1.0000002
0.30	1.0000162	1.0000003	1.0000003
0.40	1.0000215	1.0000004	1.0000004
0.50	1.0000269	1.0000006	1.0000004
0.60	1.0000322	1.0000007	1.0000005
0.70	1.0000376	1.0000008	1.0000006
0.80	1.0000430	1.0000009	1.0000007
0.90	1.0000474	1.0000010	1.0000008
0.91	1.0000451	1.0000010	1.0000008
0.92	1.0000342	1.0000010	1.0000008
0.93	0.9999889	1.0000010	1.0000008
0.94	0.9998063	1.0000010	1.0000008
0.95	0.9990744	1.0000001	1.0000008
0.96	0.9961451	0.9999857	0.9999997
0.97	0.9844262	0.9997569	0.9999673
0.98	0.9375492	0.9960947	0.9989602
0.99	0.7500396	0.9375009	0.9677427
1.00	0.0000000	0.0000000	0.0000000

Table 10.3

Numerical results for Problem 10.2, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	ε y(x)	3ε y(x)	5ε y(x)
-1.00	1.0000000	1.0000000	1.0000000
-0.90	0.8999992	0.8999998	0.9000000
-0.70	0.6999993	0.6999996	0.7000000
-0.50	0.4999964	0.4999996	0.5000000
-0.30	0.2999967	0.2999995	0.3000000
-0.10	0.1032475	0.1001375	0.1000160
-0.08	0.0866667	0.0805957	0.0801176
-0.06	0.0727863	0.0622006	0.0606941
-0.04	0.0629293	0.0468978	0.0432592
-0.02	0.0585014	0.0382678	0.0319909
0.00	0.0606750	0.0408542	0.0342021
0.02	0.0785011	0.0582681	0.0519913
0.04	0.1029288	0.0868982	0.0832598
0.06	0.1327857	0.1222010	0.1206948
0.08	0.1666660	0.1605961	0.1601183
0.10	0.2032466	0.2001379	0.2000167
0.30	0.5999947	0.5999995	0.6000006
0.50	0.9999939	0.9999994	1.0000005
0.70	1.3999956	1.3999995	1.4000003
0.90	1.7999980	1.7999997	1.8000001
1.00	2.0000000	2.0000000	2.0000000

Table 10.4

Numerical results for Problem 10.2, $\varepsilon = 10^{-4}$ and $h = 0.01$

δ → x	ε y(x)	3ε y(x)	5ε y(x)
-1.00	1.0000000	1.0000000	1.0000000
-0.90	0.8999949	0.8999997	0.9000000
-0.70	0.6999869	0.7000010	0.7000002
-0.50	0.4999862	0.5000013	0.5000003
-0.30	0.2999888	0.3000019	0.3000000
-0.10	0.1011440	0.1000631	0.1000080
-0.08	0.0823567	0.0802687	0.0800581
-0.06	0.0645274	0.0609897	0.0603426
-0.04	0.0481252	0.0430999	0.0416086
-0.02	0.0336481	0.0282083	0.0259183
0.00	0.0215117	0.0183565	0.0168811
0.02	0.0536457	0.0482087	0.0459187
0.04	0.0881205	0.0831005	0.0816091
0.06	0.1245216	0.1209904	0.1203432
0.08	0.1623499	0.1602695	0.1600586
0.10	0.2011346	0.2000638	0.2000085
0.30	0.5999607	0.6000034	0.6000012
0.50	0.9999656	1.0000030	1.0000012
0.70	1.3999826	1.4000015	1.4000008
0.90	1.7999879	1.8000000	1.8000002
1.00	2.0000000	2.0000000	2.0000000

Table 10.5

Numerical results for Problem 10.3, $\varepsilon = 10^{-3}$ and $h = 0.01$

$\delta \rightarrow$ x	ε y(x)	3ε y(x)	5ε y(x)
-1.00	1.0000000	1.0000000	1.0000000
-0.98	0.0639312	0.0102757	0.0040135
-0.96	0.0042141	0.0001095	0.0000167
-0.94	0.0002865	0.0000012	0.0000000
-0.92	0.0000201	0.0000000	0.0000000
-0.90	0.0000014	0.0000000	0.0000000
-0.70	0.0000000	0.0000000	0.0000000
-0.50	0.0000000	0.0000000	0.0000000
-0.30	0.0000000	0.0000000	0.0000000
-0.10	0.0000000	0.0000000	0.0000000
0.00	0.0000000	0.0000000	0.0000000
0.10	0.0000000	0.0000000	0.0000000
0.30	0.0000000	0.0000000	0.0000000
0.50	0.0000000	0.0000000	0.0000000
0.70	0.0000000	0.0000000	0.0000000
0.90	0.0000029	0.0000000	0.0000000
0.92	0.0000402	0.0000000	0.0000000
0.94	0.0005730	0.0000024	0.0000001
0.96	0.0084281	0.0002191	0.0000335
0.98	0.1278624	0.0205515	0.0080371
1.00	2.0000000	2.0000000	2.0000000

Table 10.6

Numerical results for Problem 10.3, $\varepsilon = 10^{-4}$ and $h = 0.01$

δ → x	ε y(x)	3ε y(x)	5ε y(x)
-1.00	1.0000000	1.0000000	1.0000000
-0.98	0.0639312	0.0102757	0.0040185
-0.96	0.0042141	0.0001095	0.0000167
-0.94	0.0002865	0.0000012	0.0000000
-0.92	0.0000201	0.0000000	0.0000000
-0.90	0.0000014	0.0000000	0.0000000
-0.70	0.0000000	0.0000000	0.0000000
-0.50	0.0000000	0.0000000	0.0000000
-0.30	0.0000000	0.0000000	0.0000000
-0.10	0.0000000	0.0000000	0.0000000
0.00	0.0000000	0.0000000	0.0000000
0.10	0.0000000	0.0000000	0.0000000
0.30	0.0000000	0.0000000	0.0000000
0.50	0.0000000	0.0000000	0.0000000
0.70	0.0000000	0.0000000	0.0000000
0.90	0.0000029	0.0000000	0.0000000
0.92	0.0000402	0.0000000	0.0000000
0.94	0.0005730	0.0000024	0.0000001
0.96	0.0084281	0.0002191	0.0000335
0.98	0.1278624	0.0205515	0.0080371
1.00	2.0000000	2.0000000	2.0000000

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